

DERIVED CATEGORIES OF FUNCTORS AND FOURIER–MUKAI TRANSFORM FOR QUIVER SHEAVES

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ABSTRACT. Let \mathcal{C} be small category and \mathcal{A} an arbitrary category. Consider the category $\mathcal{C}(\mathcal{A})$ whose objects are functors from \mathcal{C} in \mathcal{A} and whose morphisms are natural transformations. Given a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ one obtains an induced functor $F_{\mathcal{C}} : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{B})$. If \mathcal{A} and \mathcal{B} are abelian categories, we have that $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{B})$ are also abelian, and one has two functors $R(F_{\mathcal{C}}) : D(\mathcal{C}(\mathcal{A})) \rightarrow D(\mathcal{C}(\mathcal{B}))$ and $(RF)_{\mathcal{C}} : \mathcal{C}(D(\mathcal{A})) \rightarrow \mathcal{C}(D(\mathcal{B}))$. The goals of this paper are 1) to find a relationship between $D(\mathcal{C}(\mathcal{A}))$ and $\mathcal{C}(D(\mathcal{A}))$; 2) to relate the functors $R(F_{\mathcal{C}})$ and $(RF)_{\mathcal{C}}$. As an application, we prove a version of Mukai's Theorem for quiver sheaves.

1. INTRODUCTION

Let \mathcal{C} be a small category and \mathcal{A} an arbitrary category. We denote by $\mathcal{C}(\mathcal{A})$ the category whose objects are the functors from \mathcal{C} to \mathcal{A} , and whose morphisms are natural transformations. It turns out that $\mathcal{C}(\mathcal{A})$ inherits many of the properties and structures present in \mathcal{A} ; for instance, if \mathcal{A} is abelian then $\mathcal{C}(\mathcal{A})$ is also abelian (see Proposition 2 below).

An important example of this situation is provided by the quiver representation. Recall that a *quiver* $Q = (Q_0, Q_1, t, h)$ is an oriented graph consisting of two sets Q_0 (*vertices*) and Q_1 (*arrows*), and maps $t : Q_1 \rightarrow Q_0$ (*tail*) and $h : Q_1 \rightarrow Q_0$ (*head*). A path in the quiver Q is a sequence of arrows $p = a_1 a_2 \dots a_n$ with $h(a_{i+1}) = t(a_i)$ for $1 \leq i < n$; each vertex $i \in Q_0$ corresponds to a trivial path e_i . With these definitions in mind, one can associate to Q a (small) category \mathcal{Q} where each vertex is seen as an object and each path connecting two vertices is seen as a morphism between them; we say that the category \mathcal{Q} is generated by the quiver Q . Objects in $\mathcal{Q}(\mathcal{A})$ are called representations of the quiver Q in the category \mathcal{A} .

It then makes sense to consider the derived category $D(\mathcal{C}(\mathcal{A}))$. Our first goal is to find a relation between the categories $D(\mathcal{C}(\mathcal{A}))$ and $\mathcal{C}(D(\mathcal{A}))$; which are easily seen not to be equivalent in general. We show that there exists a functor $T : D(\mathcal{C}(\mathcal{A})) \rightarrow \mathcal{C}(D(\mathcal{A}))$ which is fully faithful when \mathcal{C} is generated by a quiver (cf. Theorem 21).

Now if $F : \mathcal{A} \rightarrow \mathcal{B}$ is a functor between arbitrary categories \mathcal{A} and \mathcal{B} , one can consider an *induced functor* $F_{\mathcal{C}} : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{B})$ which takes G in $\mathcal{C}(\mathcal{A})$ to the composition $F \circ G$ in $\mathcal{C}(\mathcal{B})$. The induced functor $F_{\mathcal{C}}$ also inherits some of the properties of F ; in particular, one can show that if \mathcal{A} and \mathcal{B} are abelian categories and F is additive and left exact, then so is $F_{\mathcal{C}}$.

Under the right conditions, it makes sense to consider two functors: the derived of the induced functor $R(F_{\mathcal{C}}) : D(\mathcal{C}(\mathcal{A})) \rightarrow D(\mathcal{C}(\mathcal{B}))$, and the functor induced by the derived functor $(RF)_{\mathcal{C}} : \mathcal{C}(D(\mathcal{A})) \rightarrow \mathcal{C}(D(\mathcal{B}))$. Our main result here is the Theorem 24, where we prove that, for a finite quiver Q , if $F : \mathcal{A} \rightarrow \mathcal{B}$ is a derived equivalence

between abelian categories, then the functor $R(F_Q) : D^*(\mathcal{Q}(\mathcal{A})) \rightarrow D^*(\mathcal{Q}(\mathcal{B}))$ (with $* = +, b$) is also an equivalence of categories.

All this is motivated by problems in algebraic geometry. Indeed, quiver sheaves (see for instance [4] and the references therein) and parabolic sheaves (see [10]) are examples of relevant algebraic geometric objects which can be described in terms of functors taking values in a category of sheaves on an algebraic variety.

Given a quiver Q , recall that Q -sheaf on an algebraic variety X is an object of the functor category $\mathcal{QC}(X) := \mathcal{Q}(\mathfrak{Coh}(X))$, where $\mathfrak{Coh}(X)$ is the category of coherent sheaves on X . (cf. e.g. [4]).

As an application, we consider the Fourier-Mukai transform for quiver bundles. More precisely, let X be an abelian variety and Y its dual; let also \mathcal{P} denote the Poincaré line bundle on the product $X \times Y$. Consider the functor $\mathcal{S} : \mathfrak{Coh}(X) \rightarrow \mathfrak{Coh}(Y)$ originally introduced by Mukai in [7], given by $\Phi(\mathcal{E}) = \pi_{Y*}(\pi_X^* \mathcal{E} \otimes \mathcal{P})$, where π_X and π_Y are the projections of $X \times Y$ onto the first and the second factors, respectively. Mukai has proved in [7] that \mathcal{S} is a derived equivalence, i.e. $R\mathcal{S} : D(X) \rightarrow D(Y)$ is an equivalence of categories. It follows from our main results, see details in Section 5, that the functor $R(\mathcal{S}_Q) : D^*(\mathcal{QC}(X)) \rightarrow D^*(\mathcal{QC}(Y))$, understood as a *Fourier-Mukai transform for Q -sheaves*, is also an equivalence.

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2. PRELIMINARY DEFINITIONS AND RESULTS

2.1. The category $\mathcal{C}(\mathcal{A})$. Recall that a category \mathcal{C} is called *small* if $Ob(\mathcal{C})$ is actually a set and not properly a class. Given a category \mathcal{A} , we denote by $\mathcal{C}(\mathcal{A})$ the category where $Ob(\mathcal{C}(\mathcal{A}))$ is the class consisting of all functor from \mathcal{C} to \mathcal{A} , and whose morphisms are the natural transformations.

Note that if \mathcal{C} is a small category, then

$$Mor(\mathcal{C}) = \bigcup_{(A,B) \in Ob(\mathcal{C}) \times Ob(\mathcal{C})} Hom_{\mathcal{C}}(A, B) \quad \text{and} \quad \prod_{(A,B) \in Ob(\mathcal{C}) \times Ob(\mathcal{C})} Hom_{\mathcal{C}}(A, B)$$

are sets. That guarantees that $Hom_{\mathcal{C}(\mathcal{A})}(F, G)$ be also a set by any $F, G \in Ob(\mathcal{C}(\mathcal{A}))$, which is one of the necessary conditions for $\mathcal{C}(\mathcal{A})$ let be a category.

The following Lemma (see [6, page 195]) will be useful later in the proof of Proposition 2 below.

Lemma 1. *Let \mathcal{A} be an abelian category and let f be a morphism in \mathcal{A} . Then f has factorization $f = m \circ e$ with m monic and e epi. Moreover, given any other factorization $f' = m' \circ e'$ with m' monic and e' epi and a commutative diagram*

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ a \downarrow & & \downarrow b \\ \bullet & \xrightarrow{f'} & \bullet \end{array}$$

there is a unique morphism k such that the following diagram commutes

$$\begin{array}{ccccc} \bullet & \xrightarrow{e} & \bullet & \xrightarrow{m} & \bullet \\ a \downarrow & & \downarrow k & & \downarrow b \\ \bullet & \xrightarrow{e'} & \bullet & \xrightarrow{m'} & \bullet \end{array}$$

Proposition 2. *Given \mathcal{A} an additive (abelian) category, $\mathcal{C}(\mathcal{A})$ is also an additive (abelian) category.*

Proof. We first prove that $\mathcal{C}(\mathcal{A})$ is additive.

(i) For any $F, G \in \text{Ob}(\mathcal{C}(\mathcal{A}))$, we need to prove that $\text{Hom}_{\mathcal{C}(\mathcal{A})}(F, G)$ is an abelian group: Let η_1 and η_2 be two morphisms in $\text{Hom}_{\mathcal{C}(\mathcal{A})}(F, G)$, i.e. $\eta_i = \{(\eta_i)_C \in \text{Hom}_{\mathcal{A}}(F(C), G(C)); C \in \text{Ob}(\mathcal{C})\}$, for $i = 1, 2$. Under these conditions, we define the group operation by:

$$\eta_1 + \eta_2 = \{(\eta_1 + \eta_2)_C = (\eta_1)_C + (\eta_2)_C; C \in \text{Ob}(\mathcal{C})\}.$$

Let us prove that $\eta_1 + \eta_2$ is a natural transformation between the functors F and G . We need to check that $(\eta_1 + \eta_2)_C \circ F(f) = G(f) \circ (\eta_1 + \eta_2)_D$ for all morphism $f : C \rightarrow D$ in \mathcal{C} . Since \mathcal{A} is additive we have:

$$\begin{aligned} (\eta_1 + \eta_2)_C \circ F(f) &= ((\eta_1)_C + (\eta_2)_C) \circ F(f) = \\ &= (\eta_1)_C \circ F(f) + (\eta_2)_C \circ F(f) = \\ &= G(f) \circ (\eta_1)_D + G(f) \circ (\eta_2)_D = \\ &= G(f) \circ ((\eta_1)_D + (\eta_2)_D) = \\ &= G(f) \circ (\eta_1 + \eta_2)_D. \end{aligned}$$

Remembering that $\text{Hom}_{\mathcal{A}}(F(C), G(C))$ is a group when \mathcal{A} is an additive category and taking

$$\bar{0} = \{\bar{0}_C = 0 \in \text{Hom}_{\mathcal{A}}(F(C), G(C)); C \in \text{Ob}(\mathcal{C})\}$$

as neutral element and

$$-\eta = \{(-\eta)_C = -(\eta_C); C \in \text{Ob}(\mathcal{C})\}$$

as the inverse element for any $\eta \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(F, G)$, we have that $\{\text{Hom}_{\mathcal{C}(\mathcal{A})}(F, G), +\}$ is a group.

In order to prove that $\circ : \text{Hom}_{\mathcal{C}(\mathcal{A})}(F, G) \times \text{Hom}_{\mathcal{C}(\mathcal{A})}(G, H) \rightarrow \text{Hom}_{\mathcal{C}(\mathcal{A})}(F, H)$ is bi-additive, just consider that $\circ : \text{Hom}_{\mathcal{A}}(F(C), G(C)) \times \text{Hom}_{\mathcal{A}}(G(C), H(C)) \rightarrow \text{Hom}_{\mathcal{A}}(F(C), H(C))$ is bi-additive for all $C \in \text{Ob}(\mathcal{C})$.

(ii) We define the object 0_{Ob} of $\mathcal{C}(\mathcal{A})$ for which $\text{Hom}_{\mathcal{C}(\mathcal{A})}(0_{\text{Ob}}, 0_{\text{Ob}})$ is the trivial group as follows:

$$\begin{array}{ccc} 0_{\text{Ob}} : & \mathcal{C} & \longrightarrow \mathcal{A} \\ & C & \longmapsto 0 \\ f \in \text{Hom}_{\mathcal{C}}(C, D) & \longmapsto & 0 \in \text{Hom}_{\mathcal{A}}(0, 0). \end{array}$$

Where, by abuse of notation, 0 is the zero object of \mathcal{A} and 0 is unique element of the trivial group $\text{Hom}_{\mathcal{A}}(0, 0)$, both found in \mathcal{A} by its additivity.

(iii) Given $F, G \in \text{Ob}(\mathcal{C}(\mathcal{A}))$ we must define the functor $F \oplus G$ and the natural transformations $i_F \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(F, F \oplus G)$, $i_G \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(G, F \oplus G)$, $p_F \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(F \oplus G, F)$ and $p_G \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(F \oplus G, G)$, such that

$$(1) \quad \begin{aligned} p_F \circ i_F &= \text{Id}_F, \quad p_G \circ i_G = \text{Id}_G, \quad i_F \circ p_F + i_G \circ p_G = \text{Id}_{F \oplus G} \\ p_G \circ i_F &= p_F \circ i_G = 0 \end{aligned}$$

Let C and D be objects in \mathcal{C} and $f \in \text{Hom}_{\mathcal{C}}(C, D)$. Since \mathcal{A} is an additive category there are morphisms $i_{F(C)}, i_{F(D)}, p_{F(C)}, p_{F(D)}, i_{G(C)}, i_{G(D)}, p_{G(C)}$ and $p_{G(D)}$ which satisfy the equations in (1), and such that the following diagrams are commutative:

$$\begin{array}{ccc}
 F(C) & & F(D) \\
 \downarrow i_{F(C)} & \searrow a_f & \uparrow p_{F(D)} \\
 F(C) \oplus G(C) & \xrightarrow{a} & F(D) \oplus G(D) \\
 \uparrow i_{G(C)} & \nearrow b_f & \downarrow p_{G(D)} \\
 G(C) & & G(D)
 \end{array}
 \quad
 \begin{array}{ccc}
 F(C) \oplus G(C) & \xrightarrow{e_f} & F(D) \oplus G(D) \\
 \searrow d_f & & \nearrow e_f \\
 & & F(D) \oplus G(D)
 \end{array}$$

with $a_f = i_{F(D)} \circ F(f)$, $b_f = i_{G(D)} \circ G(f)$, $e_f = F(f) \circ p_{F(C)}$, $d_f = G(f) \circ p_{G(C)}$ and a and d are the unique morphism making the diagrams commute.

Notice that $a = d$. Indeed, by the diagrams above we have

$$\begin{aligned}
 a \circ i_{F(C)} &= i_{F(D)} \circ F(f), \\
 a \circ i_{G(C)} &= i_{G(D)} \circ G(f), \\
 p_{F(D)} \circ d &= F(f) \circ p_{F(C)}, \\
 p_{G(D)} \circ d &= G(f) \circ p_{G(C)}.
 \end{aligned}
 \tag{2}$$

Composing the first line with $p_{F(C)}$, the second line with $p_{G(C)}$ and adding one to the other we have, using the equations in (1),

$$a = i_{F(D)} \circ F(f) \circ p_{F(C)} + i_{G(D)} \circ G(f) \circ p_{G(C)}.$$

Analogously, composing the third line with $i_{F(D)}$, the fourth line with $i_{G(D)}$ and adding one to the other we have

$$d = i_{F(D)} \circ F(f) \circ p_{F(C)} + i_{G(D)} \circ G(f) \circ p_{G(C)}.$$

Thus $a = d$, as desired.

Therefore we can define the functor:

$$\begin{array}{ccc}
 F \oplus G : & \mathcal{C} & \longrightarrow \mathcal{A} \\
 & C & \longmapsto (F \oplus G)(C) := F(C) \oplus G(C) \\
 & C & \longmapsto (F \oplus G)(C) \\
 & \downarrow f & \downarrow F \oplus G(f) = a = d \\
 & D & (F \oplus G)(D)
 \end{array}$$

The natural transformation $i_F \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(F, F \oplus G)$ is defined by $i_F = \{(i_F)_C = i_{F(C)} \in \text{Hom}_{\mathcal{A}}(F(C), (F \oplus G)(C)); C \in \mathcal{C}\}$. Similarly, we define the natural transformations $i_G \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(G, F \oplus G)$, $p_F \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(F \oplus G, F)$ and $p_G \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(F \oplus G, G)$, where (1) are satisfied by the way were defined the morphisms i_F, i_G, p_F and p_G .

It follows that $\{F \oplus G, p_F, p_G\}$ is the product of F and G , while $\{F \oplus G, i_F, i_G\}$ is the sum of F and G .

Therefore, $\mathcal{C}(\mathcal{A})$ is an additive category whenever \mathcal{A} is additive.

Next, we prove that if \mathcal{A} is an abelian category, then $\mathcal{C}(\mathcal{A})$ is also an abelian category.

AB 1: We need to prove that given a morphism $\eta \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(F, G)$, it has kernel and cokernel. We will show that any morphism has kernel. The argument for the existence of the cokernel is analogous.

First we must say who is the candidate to kernel of η . For each C object of \mathcal{C} we have a morphism in \mathcal{A} , $\eta_C \in \text{Hom}_{\mathcal{A}}(F(C), G(C))$, and as \mathcal{A} is abelian η_C has a kernel (K_C, i_C) . So given $f \in \text{Hom}_{\mathcal{C}}(C, D)$ we have the following commutative diagram:

$$(3) \quad \begin{array}{ccc} K_C & \xrightarrow{K_f} & K_D \\ i_C \downarrow & & \downarrow i_D \\ F(C) & \xrightarrow{F(f)} & F(D) \\ \eta_C \downarrow & & \downarrow \eta_D \\ G(C) & \xrightarrow{G(f)} & G(D) \end{array}$$

In order to guarantee the existence and uniqueness of K_f in the diagram above, consider the diagram $\eta_D \circ F(f) \circ i_C = G(f) \circ \eta_C \circ i_C$ and using that (K_C, i_C) is the kernel of η_C we know that $\eta_C \circ i_C = 0$ then $F(f) \circ i_C \in \text{Ker}((\eta_D)^*)$. Now, since (K_D, i_D) is the kernel of η_D , the sequence

$$0 \longrightarrow \text{Hom}_{\mathcal{A}}(K_C, K_D) \xrightarrow{(i_D)^*} \text{Hom}_{\mathcal{A}}(K_C, F(D)) \xrightarrow{(\eta_D)^*} \text{Hom}_{\mathcal{A}}(K_C, G(D))$$

is exact. Therefore, $\text{Ker}(\eta_D^*) = \text{Im}((i_D)^*)$, hence there is $K_f \in \text{Hom}(K_C, K_D)$ such that $i_D \circ K_f = F(f) \circ i_C$. In addition, using once again the exactness of the sequence, we obtain that $(i_D)^*$ is injective and thus conclude the uniqueness of K_f .

Therefore, the following functor is well-defined:

$$K : \begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{A} \\ C & \longmapsto & K(C) := K_C \\ C & \longmapsto & K(C) \\ f \downarrow & & \downarrow K(f) := K_f \\ D & & K(D) \end{array}$$

Clearly, $i = \{i_C; C \in \mathcal{C}\}$ is a natural transformation between K e F , since

$$\begin{array}{ccc} K(C) & \xrightarrow{i_C} & F(C) \\ K(f) \downarrow & & \downarrow F(f) \\ K(D) & \xrightarrow{i_D} & F(D) \end{array}$$

is commutative. We now check that (K, i) is the kernel of η .

Indeed, given $M \in \text{Ob}(\mathcal{C}(\mathcal{A}))$ we want to prove that the sequence

$$0 \longrightarrow \text{Hom}_{\mathcal{C}(\mathcal{A})}(M, K) \xrightarrow{i^*} \text{Hom}_{\mathcal{C}(\mathcal{A})}(M, F) \xrightarrow{\eta^*} \text{Hom}_{\mathcal{C}(\mathcal{A})}(M, G)$$

is exact, i. e., $\text{Ker}(i^*) = 0$ and $\text{Im}(i^*) = \text{Ker}(\eta^*)$.

Taking $\varphi \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(M, K)$ such that $i^*(\varphi) = 0$ then $i \circ \varphi = 0$ and therefore, for all $C \in \text{Ob}(\mathcal{C})$, we have $(i \circ \varphi)_C = 0$, or $i_C \circ \varphi_C = 0$, which means $\varphi_C \in \text{Ker}((i_C)^*)$. But, for each $C \in \text{Ob}(\mathcal{C})$ that the sequence

$$(4) \quad 0 \rightarrow \text{Hom}_{\mathcal{A}}(M(C), K(C)) \xrightarrow{(i_C)^*} \text{Hom}_{\mathcal{A}}(M(C), F(C)) \xrightarrow{(\eta_C)^*} \text{Hom}_{\mathcal{A}}(M(C), G(C))$$

is exact, so $\text{Ker}((i_C)^*) = 0$, and therefore $\varphi_C = 0$ for all object C in \mathcal{C} , thus $\varphi = 0$ and $\text{Ker}(i^*) = 0$.

We now prove that $\text{Im}(i^*) = \text{Ker}(\eta^*)$:

Consider the morphism $\alpha \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(M, F)$ such that $\alpha \in \text{Im}(i^*)$. Then there is $\alpha' \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(M, K)$ such that $\alpha = i \circ \alpha'$ so $\alpha_C = i_C \circ \alpha'_C$. By (4) we have $\text{Im}((i_C)^*) = \text{Ker}((\eta_C)^*)$ for all $C \in \text{Ob}(\mathcal{C})$, therefore $\eta_C \circ \alpha_C = 0$ for all C so $\eta \circ \alpha = 0$ and then $\alpha \in \text{Ker}(\eta^*)$. Conversely, taking $\alpha \in \text{Ker}(\eta^*)$, $\eta \circ \alpha = 0$ which implies that $\eta_C \circ \alpha_C = 0$ and then $\alpha_C \in \text{Ker}((\eta_C)^*)$, for all $C \in \text{Ob}(\mathcal{C})$. Again by the exactness of (4) there is $\alpha'_C \in \text{Hom}_{\mathcal{A}}(M_C, K_C)$ such that $\alpha_C = i_C \circ \alpha'_C$. Assuming $\alpha = \{\alpha_C; C \in \text{Ob}(\mathcal{C})\}$, we have $\alpha = i \circ \alpha'$. The proof that α is a morphism in $\mathcal{C}(\mathcal{A})$ follow of the fact that i and α' are morphisms in this category.

AB2: Let F and G objects in $\mathcal{C}(\mathcal{A})$ and let $\eta \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(F, G)$ be a monomorphism. We want to prove that η is the kernel of its cokernel. In other words, if (W, ρ) is cokernel of η we want to prove that (F, η) is the kernel of ρ :

For all $M \in \text{Ob}(\mathcal{C}(\mathcal{A}))$ we will prove that the sequence

$$(5) \quad 0 \longrightarrow \text{Hom}_{\mathcal{C}(\mathcal{A})}(M, F) \xrightarrow{\eta^*} \text{Hom}_{\mathcal{C}(\mathcal{A})}(M, G) \xrightarrow{\rho^*} \text{Hom}_{\mathcal{C}(\mathcal{A})}(M, W).$$

is exact.

Note that for each $C \in \text{Ob}(\mathcal{C})$ the sequence

$$0 \longrightarrow \text{Hom}_{\mathcal{A}}(M_C, F_C) \xrightarrow{(\eta_C)^*} \text{Hom}_{\mathcal{A}}(M_C, G_C) \xrightarrow{(\rho_C)^*} \text{Hom}_{\mathcal{A}}(M_C, W_C)$$

is exact, so the sequence (5) is also exact.

AB 3: Analogous to **AB 2**.

AB 4: We need to show that every morphism is the composition of an epimorphism with a monomorphism.

Let F and G be objects in $\mathcal{C}(\mathcal{A})$ and take $\eta \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(F, G)$. As \mathcal{A} is abelian, for all $C \in \text{Ob}(\mathcal{C})$, $\eta_C \in \text{Hom}_{\mathcal{A}}(F(C), G(C))$ can be written by $\eta_C = \beta_C \circ \alpha_C$, where $\alpha_C \in \text{Hom}_{\mathcal{A}}(F(C), H(C))$ is an epimorphism and $\beta_C \in \text{Hom}_{\mathcal{A}}(H(C), G(C))$ is a monomorphism. We know that for $f \in \text{Hom}_{\mathcal{C}}(C, D)$ the diagram

$$\begin{array}{ccc} F(C) & \xrightarrow{\eta_C} & G(C) \\ F(f) \downarrow & & \downarrow G(f) \\ F(D) & \xrightarrow{\eta_D} & G(D) \end{array}$$

is commutative and, by Lemma 1, for each object C in \mathcal{C} there is unique $H(f) \in \text{Hom}(H(C), H(D))$ such that the following diagram is commutative:

$$\begin{array}{ccccc}
 F(C) & \xrightarrow{\alpha_C} & H(C) & \xrightarrow{\beta_C} & G(C) \\
 F(f) \downarrow & & \downarrow H(f) & & \downarrow G(f) \\
 F(D) & \xrightarrow{\alpha_D} & H(D) & \xrightarrow{\beta_D} & G(D)
 \end{array}$$

Therefore, we have that H is a functor, $\alpha = \{\alpha_C; C \in \text{Ob}(\mathcal{C})\}$ and $\beta = \{\beta_C; C \in \text{Ob}(\mathcal{C})\}$ are natural transformations such that $\alpha \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(F, H)$ is an epimorphism, and $\beta \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(H, G)$ is monomorphism and $\eta = \beta \circ \alpha$.

This concludes the proof that $\mathcal{C}(\mathcal{A})$ is an abelian category. \square

Recall that an abelian category \mathcal{A} is said to be *complete* if the product of any family of objects exists in \mathcal{A} ; that is, given a family $\{A_j\}_{j \in J}$ of objects of \mathcal{A} , the product $\prod_{j \in J} A_j$ is an object of \mathcal{A} .

Lemma 3. *If \mathcal{A} be a complete abelian category, then so is $\mathcal{C}(\mathcal{A})$.*

The proof of this result is analogous to the proof of property (iii) of additivity for $\mathcal{C}(\mathcal{A})$.

The following Proposition provides a sufficient condition that guarantees that the functor category $\mathcal{C}(\mathcal{A})$ has enough injectives; it is Exercise 2.3.13 in [9, page 43].

Proposition 4. *If \mathcal{A} is a complete abelian category with enough injectives then $\mathcal{C}(\mathcal{A})$ also has enough injectives.*

Corollary 5. *Let \mathcal{A} be an abelian category with enough injectives and let \mathcal{C} be a category with a finite number of objects and morphisms. Then $\mathcal{C}(\mathcal{A})$ has enough injectives.*

2.2. \mathcal{A} as a full subcategory of $\mathcal{C}(\mathcal{A})$. Let \mathcal{A} be an abelian category. Set $D \in \text{Ob}(\mathcal{C})$ and $A \in \text{Ob}(\mathcal{A})$, and consider the following functor $I_D(A)$ in $\mathcal{C}(\mathcal{A})$:

$$I_D(A)(C) = \begin{cases} 0 & \text{se } C \neq D \\ A & \text{se } C = D; \end{cases}$$

and given $t \in \text{Hom}_{\mathcal{C}}(C, D)$:

$$I_D(A)(t) = \begin{cases} Id_A & \text{se } t = Id_D \\ 0 & \text{caso contrário.} \end{cases}$$

Taking $f \in \text{Hom}_{\mathcal{A}}(A_1, A_2)$, we can define a natural transformation $\varphi = \{\varphi_C; C \in \text{Ob}(\mathcal{C})\}$ from $I_D(A_1)$ to $I_D(A_2)$ in the following manner:

$$\varphi_C = \begin{cases} f & \text{if } C = D \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have the functor:

$$\begin{array}{ccc}
 I_D : & \mathcal{A} & \longrightarrow & \mathcal{C}(\mathcal{A}) \\
 & A_1 & \longmapsto & I_D(A_1) \\
 & \downarrow f & & \downarrow \varphi \\
 & A_2 & & I_D(A_2).
 \end{array}$$

Proposition 6. *The functor I_D is full and faithful for each $D \in \text{Ob}(\mathcal{C})$.*

Proof. Let $A_1, A_2 \in \text{Ob}(\mathcal{A})$. Any morphism $f : A_1 \rightarrow A_2$ generates an unique natural transformation φ as defined above. On the other hand, since $\text{Hom}_{\mathcal{A}}(0, 0)$, $\text{Hom}_{\mathcal{A}}(0, A)$ and $\text{Hom}_{\mathcal{A}}(A, 0)$ are trivial groups for any $A \in \text{Ob}(\mathcal{A})$, the choice of a natural transformation between $I_D(A_1)$ and $I_D(A_2)$, determines an unique morphism in $\text{Hom}_{\mathcal{A}}(A_1, A_2)$. It follows that

$$\text{Hom}_{\mathcal{A}}(A_1, A_2) \simeq \text{Hom}_{\mathcal{C}(\mathcal{A})}(I_D(A_1), I_D(A_2)),$$

as desired. \square

Proposition 7. *The functor I_D is an exact functor for each $D \in \text{Ob}(\mathcal{C})$.*

Proof. Given an exact sequence in \mathcal{A} , $0 \rightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \rightarrow 0$, we need to prove that $0 \rightarrow I_D(A') \xrightarrow{I_D(f)} I_D(A) \xrightarrow{I_D(g)} I_D(A'') \rightarrow 0$ is exact in $\mathcal{C}(\mathcal{A})$.

Indeed, $I_D(f)$ is monomorphism whenever, for each $C \in \text{Ob}(\mathcal{C})$, $I_D(f)_C$ is monomorphism. However, $I_D(f)_D = f$ and f is monomorphism. In an analogous way, we have that $I_D(g)$ is epimorphism. Moreover, as $\text{Ker}(g) = \text{Im}(f)$ we have that $\text{Ker}(I_D(g)) = \text{Im}(I_D(f))$, thus $\text{Ker}(I_D(g))_C = \text{Im}(I_D(f))_C$ for each $C \in \text{Ob}(\mathcal{C})$. \square

2.3. The induced functor. Any functor between categories \mathcal{A} and \mathcal{B} induces in a natural way a functor between $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{B})$ which inherits some of the properties of the original functor. More precisely, consider the following definition.

Definition 8. *Let \mathcal{A} and \mathcal{B} be categories and let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a functor. The induced functor $F_{\mathcal{C}} : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{B})$ is defined by:*

$$\begin{array}{ccc} F_{\mathcal{C}} : \mathcal{C}(\mathcal{A}) & \longrightarrow & \mathcal{C}(\mathcal{B}) \\ G & \longmapsto & F_{\mathcal{C}}(G) := F \circ G \\ G & \longmapsto & F \circ G \\ \eta \downarrow & & \downarrow F_{\mathcal{C}}(\eta) \\ H & & F \circ H \end{array}$$

Where $F_{\mathcal{C}}(\eta) = \{(F_{\mathcal{C}}(\eta))_C := F(\eta_C) \in \text{Hom}_{\mathcal{B}}(F(G(C)), F(H(C))) ; C \in \text{Ob}(\mathcal{C})\}$.

Let us now see what properties $F_{\mathcal{C}}$ inherits from F .

Proposition 9. *Let \mathcal{A} and \mathcal{B} be additive categories and let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor. Then the induced functor $F_{\mathcal{C}}$ is also additive.*

Proof. Set $\alpha, \beta \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(R, S)$, we need to prove that $F_{\mathcal{C}}(\alpha + \beta) = F_{\mathcal{C}}(\alpha) + F_{\mathcal{C}}(\beta)$. By definition,

$$F_{\mathcal{C}}(\alpha + \beta) = \{F(\alpha_C + \beta_C) \in \text{Hom}_{\mathcal{C}(\mathcal{B})}(F(R(C)), F(S(C))) : C \in \text{Ob}(\mathcal{C})\},$$

Since F is additive, $F(\alpha_C + \beta_C) = F(\alpha_C) + F(\beta_C)$ and

$$\begin{aligned} F_{\mathcal{C}}(\alpha + \beta) &= \{F(\alpha_C) + F(\beta_C) : C \in \text{Ob}(\mathcal{C})\} \\ &= \{F(\alpha_C) : C \in \text{Ob}(\mathcal{C})\} \cup \{F(\beta_C) : C \in \text{Ob}(\mathcal{C})\} \\ &= F_{\mathcal{C}}(\alpha) + F_{\mathcal{C}}(\beta). \end{aligned}$$

\square

Proposition 10. *Let \mathcal{A} and \mathcal{B} be abelian categories and let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a functor. F is exact if, and only if, the induced functor $F_{\mathcal{C}}$ is exact.*

Proof. Let

$$(6) \quad 0 \longrightarrow R' \xrightarrow{\eta} R \xrightarrow{\xi} R'' \longrightarrow 0$$

be an exact sequence in $\mathcal{C}(\mathcal{A})$. We must to prove that

$$(7) \quad 0 \longrightarrow F_{\mathcal{C}}(R') \xrightarrow{F_{\mathcal{C}}(\eta)} F_{\mathcal{C}}(R) \xrightarrow{F_{\mathcal{C}}(\xi)} F_{\mathcal{C}}(R'') \longrightarrow 0$$

is an exact sequence in $\mathcal{C}(\mathcal{B})$.

Note that (6) is exact if, and only if, for each object C de \mathcal{C} , the sequence

$$0 \longrightarrow R'(C) \xrightarrow{\eta_C} R(C) \xrightarrow{\xi_C} R''(C) \longrightarrow 0$$

is also exact in \mathcal{A} . Therefore since F is an exact functor, the sequence

$$0 \longrightarrow F(R'(C)) \xrightarrow{F(\eta_C)} F(R(C)) \xrightarrow{F(\xi_C)} F(R''(C)) \longrightarrow 0$$

is exact in \mathcal{B} . However, by definition of $F_{\mathcal{C}}$, $F(R'(C)) = F_{\mathcal{C}}R'(C)$, $F(\eta_C) = (F_{\mathcal{C}}(\eta))_C$, and analogously for R , R'' and ξ . It follows that, for each $C \in \text{Ob}(\mathcal{C})$, the sequence

$$0 \longrightarrow F_{\mathcal{C}}R'(C) \xrightarrow{(F_{\mathcal{C}}(\eta))_C} F_{\mathcal{C}}R(C) \xrightarrow{(F_{\mathcal{C}}(\xi))_C} F_{\mathcal{C}}R''(C) \longrightarrow 0$$

is exact in \mathcal{B} so (7) is exact in $\mathcal{C}(\mathcal{B})$.

Conversely, let $0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$ be an exact sequence in \mathcal{A} . By Proposition 7, we have that I_D is an exact functor for all $D \in \text{Ob}(\mathcal{C})$, then the sequence

$$0 \longrightarrow I_D(A') \xrightarrow{I_D(f)} I_D(A) \xrightarrow{I_D(g)} I_D(A'') \longrightarrow 0 \text{ is exact in } \mathcal{C}(\mathcal{A}).$$

By hypothesis, $F_{\mathcal{C}}$ is an exact functor and

$$\begin{aligned} F_{\mathcal{C}}(I_D(A)) &= F \circ I_D(A) = I_D(F(A)) \quad \text{and} \\ F_{\mathcal{C}}(I_D(f)) &= F \circ I_D(f) = I_D(F(f)), \end{aligned}$$

for all $A \in \text{Ob}(\mathcal{A})$, thus the sequence

$$0 \longrightarrow I_D(F(A')) \xrightarrow{I_D(F(f))} I_D(F(A)) \xrightarrow{I_D(F(g))} I_D(F(A'')) \longrightarrow 0$$

is exact in $\mathcal{C}(\mathcal{B})$. Therefore, for each $C \in \text{Ob}(\mathcal{C})$ the sequence

$$0 \rightarrow (I_D(F(A')))_C \xrightarrow{(I_D(F(f)))_C} (I_D(F(A)))_C \xrightarrow{(I_D(F(g)))_C} (I_D(F(A'')))_C \rightarrow 0$$

is exact in \mathcal{B} . In particular, if $C = D$ we have that the sequence

$$0 \longrightarrow F(A') \xrightarrow{F(f)} F(A) \xrightarrow{F(g)} F(A'') \longrightarrow 0$$

is exact in \mathcal{B} . Hence F is an exact functor. \square

Proposition 11. *Let \mathcal{A} and \mathcal{B} be categories and let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an equivalence. Then $F_{\mathcal{C}} : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{B})$ is an equivalence.*

Proof. We first check that $F_{\mathcal{C}}$ is full and faithful; given two objects R_1 and R_2 of $\mathcal{C}(\mathcal{A})$, we must prove that the map

$$F_{\mathcal{C}} : \text{Hom}_{\mathcal{C}(\mathcal{A})}(R_1, R_2) \longrightarrow \text{Hom}_{\mathcal{C}(\mathcal{B})}(F_{\mathcal{C}}(R_1), F_{\mathcal{C}}(R_2))$$

is bijective.

Set $\beta \in \text{Hom}_{\mathcal{C}(\mathcal{B})}(F_{\mathcal{C}}(R_1), F_{\mathcal{C}}(R_2))$. Then, for each object C of \mathcal{C} , we have that $\beta_C \in \text{Hom}_{\mathcal{B}}(F(R_1(C)), F(R_2(C)))$. Since F is full, there is a morphism $\alpha_C : R_1(C) \rightarrow R_2(C)$ such that $F(\alpha_C) = \beta_C$.

Taking $\alpha = \{\alpha_C; C \in \text{Ob}(\mathcal{C})\}$ we will prove that α is a natural transformation between R_1 and R_2 , that is, given a morphism $t : C \rightarrow D$ in \mathcal{C} ,

$$(8) \quad R_2(t) \circ \alpha_C = \alpha_D \circ R_1(t).$$

Since $\beta_C = F(\alpha_C)$, and using that β is a natural transformation, for each $t : C \rightarrow D$ in \mathcal{C} , we have that the diagram

$$\begin{array}{ccc} F(R_1(C)) & \xrightarrow{F(\alpha_C)} & F(R_2(C)) \\ F(R_1(t)) \downarrow & & \downarrow F(R_2(t)) \\ F(R_1(D)) & \xrightarrow{F(\alpha_D)} & F(R_2(D)) \end{array}$$

is commutative, that is,

$$F(R_2(t) \circ \alpha_C) = F(\alpha_D \circ R_1(t)).$$

Since F is faithful, (8) is true. This shows that $F_{\mathcal{C}}$ is also full.

Given $\alpha^{(1)}$ and $\alpha^{(2)}$ in $\text{Hom}_{\mathcal{C}(\mathcal{A})}(R_1, R_2)$ such that $F_{\mathcal{C}}(\alpha^{(1)}) = F_{\mathcal{C}}(\alpha^{(2)})$. For each object C of \mathcal{C} , we have $(F_{\mathcal{C}}(\alpha^{(1)}))_C = (F_{\mathcal{C}}(\alpha^{(2)}))_C$, that is, $F(\alpha_C^{(1)}) = F(\alpha_C^{(2)})$. Since F is faithful $\alpha_C^{(1)} = \alpha_C^{(2)}$ for each C , then $\alpha^{(1)} = \alpha^{(2)}$. Hence $F_{\mathcal{C}}$ is faithful.

Next, we show that the induced functor is essentially surjective. Let S be an object of $\mathcal{C}(\mathcal{B})$. We must show that there is R in $\text{Ob}(\mathcal{C}(\mathcal{A}))$ such that $F_{\mathcal{C}}(R) \simeq S$, that is, there exist a natural isomorphism between $F \circ R$ and S .

Since F is an equivalence, for each C in $\text{Ob}(\mathcal{C})$, exists an object R_C of \mathcal{A} such that $F(R_C) \simeq S(C)$ in \mathcal{B} . Then there exists at least one isomorphism $\eta_C \in \text{Hom}_{\mathcal{B}}(F(R_C), S(C))$. Set η_C for each C in $\text{Ob}(\mathcal{C})$; hence, for each morphism $t : C \rightarrow D$ in \mathcal{C} , we have the following isomorphism:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{B}}(S(C), S(D)) & \simeq & \text{Hom}_{\mathcal{B}}(F(R_C), F(R_D)) \\ S(t) & \mapsto & \eta_D^{-1} \circ S(t) \circ \eta_C. \end{array}$$

Then, because F is full and faithful, there exist an unique morphism R_t in $\text{Hom}_{\mathcal{B}}(R_C, R_D)$ such that $F(R_t) = \eta_D^{-1} \circ S(t) \circ \eta_C$.

Therefore R is a functor from \mathcal{C} to \mathcal{A} , and the following diagram is commutative:

$$\begin{array}{ccc} F(R(C)) & \xrightarrow{\eta_C} & S(C) \\ F(R(t)) \downarrow & & \downarrow S(t) \\ F(R(D)) & \xrightarrow{\eta_D} & S(D), \end{array}$$

proving that η is a natural isomorphism between $F \circ R$ and S . \square

With mild additional hypotheses, the converse is also true.

Proposition 12. *Let \mathcal{A} and \mathcal{B} be additive categories and let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor. Then F is an equivalence if, and only if, $F_{\mathcal{C}}$ is also an equivalence.*

Proof. The first implication follows of the previous Proposition. To establish the converse statement, let $F_{\mathcal{C}} : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{C}(\mathcal{B})$ be an equivalence of categories.

Given $B \in \text{Ob}(\mathcal{B})$ and $D \in \text{Ob}(\mathcal{C})$, consider the functor $I_D(B) : \mathcal{C} \longrightarrow \mathcal{B}$ defined above. Since $F_{\mathcal{C}}$ is an equivalence, there exists $K \in \text{Ob}(\mathcal{C}(\mathcal{A}))$ such that $F_{\mathcal{C}}(K) \simeq I_D(B)$, that is, $F \circ K \simeq I_D(B)$. Then there is a natural isomorphism η between $F \circ K$ and $I_D(B)$. Thus there exists an isomorphism $\eta_D \in \text{Hom}_{\mathcal{B}}(F \circ K(D), I_D(B)(D))$, hence $F(K(D)) \simeq I_D(B)(D) = B$ and $K(D) \in \text{Ob}(\mathcal{A})$.

Let A_1 and A_2 objects of \mathcal{A} . As I_D is full and faithful, in Proposition 6, we have

$$\text{Hom}_{\mathcal{A}}(A_1, A_2) \simeq \text{Hom}_{\mathcal{C}(\mathcal{A})}(I_D(A_1), I_D(A_2)),$$

then, as $F_{\mathcal{C}}$ is an equivalence

$$\text{Hom}_{\mathcal{C}(\mathcal{A})}(I_D(A_1), I_D(A_2)) \simeq \text{Hom}_{\mathcal{C}(\mathcal{B})}(F_{\mathcal{C}}(I_D(A_1)), F_{\mathcal{C}}(I_D(A_2))).$$

However, $F_{\mathcal{C}}(I_D(A_i)) = F \circ I_D(A_i) = I_D(F(A_i))$, hence

$$\begin{aligned} \text{Hom}_{\mathcal{C}(\mathcal{B})}(F_{\mathcal{C}}(I_D(A_1)), F_{\mathcal{C}}(I_D(A_2))) &\simeq \text{Hom}_{\mathcal{C}(\mathcal{B})}(I_D(F(A_1)), I_D(F(A_2))) \\ &\simeq \text{Hom}_{\mathcal{B}}(F(A_1), F(A_2)) \end{aligned}$$

It follows that

$$\text{Hom}_{\mathcal{A}}(A_1, A_2) \simeq \text{Hom}_{\mathcal{B}}(F(A_1), F(A_2)),$$

as desired. \square

3. COMPARISON BETWEEN $D(\mathcal{C}(\mathcal{A}))$ AND $\mathcal{C}(D(\mathcal{A}))$

3.1. The isomorphism between $\text{Kom}(\mathcal{C}(\mathcal{A}))$ and $\mathcal{C}(\text{Kom}(\mathcal{A}))$. An object (F^{\bullet}, d_F) in $\text{Kom}(\mathcal{C}(\mathcal{A}))$ is a complex in the functor category $\mathcal{C}(\mathcal{A})$, that is, is a complex

$$\dots \longrightarrow F^{n-1} \xrightarrow{d_F^{n-1}} F^n \xrightarrow{d_F^n} F^{n+1} \longrightarrow \dots$$

where $F^i : \mathcal{C} \longrightarrow \mathcal{A}$ is a functor, and d_F^i is a natural transformation between F^i and F^{i+1} , for each $i \in \mathbb{Z}$. More precisely,

$$d_F^i = \{(d_F^i)_C \in \text{Hom}_{\mathcal{A}}(F^i(C), F^{i+1}(C)); C \in \text{Ob}(\mathcal{C})\},$$

where, given a morphism $C \xrightarrow{t} D$ in \mathcal{C} we have $F^{i+1}(t) \circ (d_F^i)_C = (d_F^i)_D \circ F^i(t)$.

Thus for each $C \in \text{Ob}(\mathcal{C})$, we have an object $(F(C)^{\bullet}, d_{F(C)})$ in $\text{Kom}(\mathcal{A})$:

$$\dots \longrightarrow F^{n-1}(C) \xrightarrow{(d_F^{n-1})_C} F^n(C) \xrightarrow{(d_F^n)_C} F^{n+1}(C) \longrightarrow \dots$$

A morphism $\eta^{\bullet} \in \text{Hom}_{\text{Kom}(\mathcal{C}(\mathcal{A}))}((F^{\bullet}, d_F), (G^{\bullet}, d_G))$ is a family of natural transformations $\eta^{\bullet} = \{\eta^i \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(F^i, G^i) / d_G^i \circ \eta^i = \eta^{i+1} \circ d_F^i; i \in \mathbb{Z}\}$. That is, for each morphism $C \xrightarrow{t} D$ in \mathcal{C} we have the following commutative cube

$$(9) \quad \begin{array}{ccccc} & & G^{i-1}(C) & & \\ & \nearrow \eta_C^{i-1} & \downarrow & \searrow (d_G^{i-1})_C & \\ F^{i-1}(C) & & & & G^i(C) \\ & \searrow (d_F^{i-1})_C & \downarrow G^{i-1}(t) & \nearrow \eta_C^i & \\ & & F^i(C) & & \\ & & \downarrow F^i(t) & & \downarrow G^i(t) \\ & & G^{i-1}(D) & & \\ & \nearrow \eta_D^{i-1} & \downarrow & \searrow (d_G^i)^{-1}_D & \\ F^{i-1}(D) & & & & G^i(D) \\ & \searrow (d_F^i)^{-1}_D & \downarrow & \nearrow \eta_D^i & \\ & & F^i(D) & & \end{array}$$

With these basic ideas in mind, we have the following Lemma:

Lemma 13. *Let \mathcal{C} be a small category and let \mathcal{A} be an abelian category. Then the categories $Kom(\mathcal{C}(\mathcal{A}))$ and $\mathcal{C}(Kom(\mathcal{A}))$ are isomorphic.*

Proof. We want to find functors $K : Kom(\mathcal{C}(\mathcal{A})) \rightarrow \mathcal{C}(Kom(\mathcal{A}))$ and $K' : \mathcal{C}(Kom(\mathcal{A})) \rightarrow Kom(\mathcal{C}(\mathcal{A}))$ such that $K \circ K' = 1_{\mathcal{C}(Kom(\mathcal{A}))}$ e $K' \circ K = 1_{Kom(\mathcal{C}(\mathcal{A}))}$.

Firstly we will define K :

Let (F^\bullet, d_F) be an object of $Kom(\mathcal{C}(\mathcal{A}))$, we take $K((F^\bullet, d_F))$ as the functor F of \mathcal{C} in $Kom(\mathcal{A})$ defined by:

$$\begin{array}{ccc} F : & \mathcal{C} & \longrightarrow & Kom(\mathcal{A}) \\ & C & \longmapsto & (F(C)^\bullet, d_{F(C)}) \\ & \downarrow t & & \downarrow F(t)^\bullet \\ & D & & (F(D)^\bullet, d_{F(D)}) \end{array}$$

where $F(t)^\bullet = \{(F(t))^i = F^i(t); i \in \mathbb{Z}\}$; note that $F(t)^\bullet$ is a morphism in $Kom(\mathcal{A})$. Indeed, since each d_F^i is a natural transformation and each F^i is a functor, the diagram

$$\begin{array}{ccc} F^i(C) & \xrightarrow{(d_F^i)_C} & F^{i+1}(C) \\ \downarrow F^i(t) & & \downarrow F^{i+1}(t) \\ F^i(D) & \xrightarrow{(d_F^i)_D} & F^{i+1}(D) \end{array}$$

is commutative for all $i \in \mathbb{Z}$ and for all $C \xrightarrow{t} D$ morphism in \mathcal{C} .

Now, given $\eta^\bullet \in \text{Hom}_{\text{Kom}(\mathcal{C}(\mathcal{A}))}((F^\bullet, d_F), (G^\bullet, d_G))$, we have:

$$\begin{aligned} \eta^\bullet &= \{\eta^i \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(F^i, G^i) / d_G^i \circ \eta^i = \eta^{i+1} \circ d_F^i; i \in \mathbb{Z}\} \\ &= \{(\eta^i)_C \in \text{Hom}_{\mathcal{A}}(F^i(C), G^i(C)) \text{ tal que} \\ &\quad (d_G^i \circ \eta^i)_C = (\eta^{i+1} \circ d_F^i)_C; i \in \mathbb{Z}, C \in \text{Ob}(\mathcal{C})\}. \end{aligned}$$

We define

$$\begin{aligned} \eta := K(\eta^\bullet) &= \{(\eta_C)^\bullet \in \text{Hom}_{\text{Kom}(\mathcal{A})}(F(C)^\bullet, G(C)^\bullet); C \in \text{Ob}(\mathcal{C})\} \\ &= \{(\eta_C)^i \in \text{Hom}_{\mathcal{A}}(F(C))^i, (G(C))^i) \text{ tal que} \\ &\quad (d_G^i \circ \eta^i)_C = (\eta^{i+1} \circ d_F^i)_C; i \in \mathbb{Z}, C \in \text{Ob}(\mathcal{C})\}. \end{aligned}$$

By the commutative cube (9), η is a natural transformation between $F = K((F^\bullet, d_F))$ and $G = K((G^\bullet, d_G))$.

In short, K is the following functor:

$$\begin{array}{ccc} K : \text{Kom}(\mathcal{C}(\mathcal{A})) & \longrightarrow & \mathcal{C}(\text{Kom}(\mathcal{A})) \\ (F^\bullet, d_F) & \longmapsto & F \\ \downarrow \eta^\bullet & & \downarrow \eta \\ (G^\bullet, d_G) & & G \end{array}$$

Let us define K' :

Take $F \in \text{Ob}(\mathcal{C}(\text{Kom}(\mathcal{A})))$. Then, for each $C \in \text{Ob}(\mathcal{C})$, $F(C)$ is a complex in \mathcal{A} and for each $t \in \text{Hom}_{\mathcal{C}}(C, D)$, $F(t)$ is a morphism of complexes:

$$\begin{array}{ccccccc} F(C) = & \dots & \longrightarrow & F(C)^{n-1} & \xrightarrow{d_{F(C)}^{n-1}} & F(C)^n & \xrightarrow{d_{F(C)}^n} & F(C)^{n+1} & \longrightarrow & \dots \\ \downarrow F(t) & & & \downarrow F(t)^{n-1} & & \downarrow F(t)^n & & \downarrow F(t)^{n+1} & & \\ F(D) = & \dots & \longrightarrow & F(D)^{n-1} & \xrightarrow{d_{F(D)}^{n-1}} & F(D)^n & \xrightarrow{d_{F(D)}^n} & F(D)^{n+1} & \longrightarrow & \dots \end{array}$$

Therefore, as F is a functor between \mathcal{C} and $\text{Kom}(\mathcal{A})$, we have that each F^i is defined by

$$\begin{array}{ccc} F^i : \mathcal{C} & \longrightarrow & \mathcal{A} \\ C & \longmapsto & F^i(C) = F(C)^i \\ \downarrow t & & \downarrow F^i(t) = F(t)^i \\ D & & F^i(D) = F(D)^i \end{array}$$

Moreover, we can define for each $i \in \mathbb{Z}$, $d_F^i = \{d_{F(C)}^i \in \text{Hom}_{\mathcal{A}}(F(C)^i, F(C)^{i+1}); C \in \text{Ob}(\mathcal{C})\}$ which, by definition of F , is a natural transformation satisfying $d_F^{i+1} \circ d_F^i = 0$, for all $i \in \mathbb{Z}$. Hence

$$\dots \longrightarrow F^{n-1} \xrightarrow{d_F^{n-1}} F^n \xrightarrow{d_F^n} F^{n+1} \longrightarrow \dots$$

is an object of $\text{Kom}(\mathcal{C}(\mathcal{A}))$ that will be the image of F by the functor K' .

Let η be a natural transformation between F and G , then

$$\begin{aligned} \eta &= \{\eta_C \in \text{Hom}_{\text{Kom}(\mathcal{A})}(F(C)^\bullet, G(C)^\bullet); C \in \text{Ob}(\mathcal{C})\} \\ &= \{(\eta_C)^i \in \text{Hom}_{\mathcal{A}}(F(C))^i, (G(C))^i) \text{ tal que} \\ &\quad (d_G^i \circ \eta^i)_C = (\eta^{i+1} \circ d_F^i)_C; i \in \mathbb{Z}, C \in \text{Ob}(\mathcal{C})\}. \end{aligned}$$

Namely, $\eta = \{(\eta_C)^\bullet \in \text{Hom}_{\text{Kom}(\mathcal{A})}(F(C)^\bullet, G(C)^\bullet); C \in \text{Ob}(\mathcal{C})\}$.

We will define

$$\begin{aligned}\eta^\bullet = K'(\eta) &= \{(\eta_C)^\bullet \in \text{Hom}_{\text{Kom}(\mathcal{A})}(F(C)^\bullet, G(C)^\bullet); C \in \text{Ob}(\mathcal{C})\} \\ &= \{(\eta_C)^i \in \text{Hom}_{\mathcal{A}}(F(C))^i, (G(C))^i\} \text{ tal que} \\ &\quad (d_G^i \circ \eta^i)_C = (\eta^{i+1} \circ d_F^i)_C; i \in \mathbb{Z}, C \in \text{Ob}(\mathcal{C})\}.\end{aligned}$$

Therefore, by the commutativity of cube (9), $K'(\eta)$ is a morphism in $\text{Kom}(\mathcal{C}(\mathcal{A}))$ between (F^\bullet, d_F) and (G^\bullet, d_G) .

In summary, K' is defined as follows:

$$\begin{array}{ccc} K' : \mathcal{C}(\text{Kom}(\mathcal{A})) & \longrightarrow & \text{Kom}(\mathcal{C}(\mathcal{A})) \\ F & \longmapsto & (F^\bullet, d_F) \\ \downarrow \eta & & \downarrow \eta^\bullet \\ G & & (G^\bullet, d_G). \end{array}$$

So, K maps $(\eta^i)_C \mapsto (\eta_C)^i$ and K' maps $(\eta_C)^i \mapsto (\eta^i)_C$. Hence K and K' restricted to Hom' 's coincide with the identity (they are only a change of indices). Furthermore, $K \circ K'$ and $K' \circ K$ act trivially on objects, so are the identity functors of $\mathcal{C}(\text{Kom}(\mathcal{A}))$ and $\text{Kom}(\mathcal{C}(\mathcal{A}))$, respectively.

We conclude that $\mathcal{C}(\text{Kom}(\mathcal{A}))$ and $\text{Kom}(\mathcal{C}(\mathcal{A}))$ are isomorphic. \square

Similarly, one can show that $\text{Kom}^*(\mathcal{C}(\mathcal{A}))$ and $\mathcal{C}(\text{Kom}^*(\mathcal{A}))$ are also isomorphic, for $*$ = +, - or b .

Let $\text{Kom}_0(\mathcal{A})$ be the category of complexes whose differentials are all zero. Then we have:

Corollary 14. *Under the same hypothesis of the last theorem, $\text{Kom}_0(\mathcal{C}(\mathcal{A}))$ and $\mathcal{C}(\text{Kom}_0(\mathcal{A}))$ are isomorphic.*

3.2. The category $D(\mathcal{C}(\mathcal{A}))$. The objects of $D(\mathcal{C}(\mathcal{A}))$ are the same objects of $\text{Kom}(\mathcal{C}(\mathcal{A}))$. Using the isomorphism K , we can think of the objects of $D(\mathcal{C}(\mathcal{A}))$ as objects of $\mathcal{C}(\text{Kom}(\mathcal{A}))$, thus for each $F^\bullet \in \text{Ob}(D(\mathcal{C}(\mathcal{A})))$ we associate a functor $F : \mathcal{C} \longrightarrow \text{Kom}(\mathcal{A})$.

A morphism $F^\bullet \longrightarrow G^\bullet$ in $D(\mathcal{C}(\mathcal{A}))$ is a class of diagrams of the form:

$$(10) \quad \begin{array}{ccc} & H^\bullet & \\ [\alpha] \swarrow & & \searrow [f] \\ F^\bullet & & G^\bullet, \end{array}$$

where $[f]$ is a morphism in $K(\mathcal{C}(\mathcal{A}))$ and $[\alpha]$ is a quasi-isomorphisms in $K(\mathcal{C}(\mathcal{A}))$.

We note that if $f \sim g$ in $\text{Kom}(\mathcal{C}(\mathcal{A}))$ then $f_C \sim g_C$ in $\text{Kom}(\mathcal{A})$, for all $C \in \text{Ob}(\mathcal{C})$. Recall also that f_C comes from the isomorphism between $\text{Kom}(\mathcal{C}(\mathcal{A}))$ and $\mathcal{C}(\text{Kom}(\mathcal{A}))$, as explained above.

Moreover, we will prove that, if α is a quasi-isomorphism in $\text{Kom}(\mathcal{C}(\mathcal{A}))$ then α_C is a quasi-isomorphism in $\text{Kom}(\mathcal{A})$. Therefore, given the diagram (10) we have, for each object $C \in \text{Ob}(\mathcal{C})$, the following diagram in $D(\mathcal{A})$:

$$\begin{array}{ccc}
& H(C) & \\
[\alpha_C] \swarrow & & \searrow [f_C] \\
F(C) & & G(C),
\end{array}$$

where $H(C)$, $F(C)$ and $G(C)$ are the objects of $Kom(\mathcal{A})$ induced by the isomorphism K described in the proof of Lemma 13. For each morphism $C \xrightarrow{t} D$ in \mathcal{C} we have the following diagram in $K(\mathcal{A})$, where each square is commutative:

$$(11) \quad
\begin{array}{ccccc}
& H(C) & \xrightarrow{H(t)} & H(D) & \\
\alpha_C \swarrow & & & & \searrow \alpha_D \\
F(C) & \xrightarrow{f_C \quad F(t)} & F(D) & & \\
\downarrow & & \downarrow f_D & & \\
G(C) & \xrightarrow{G(t)} & G(D) & &
\end{array}$$

Let us now prove that a quasi-isomorphism α in $Kom(\mathcal{C}(\mathcal{A}))$ induces a quasi-isomorphism α_C in $Kom(\mathcal{A})$. For this, we require the following Lemma, whose proof can be found in [8, Corollary 2.11.9, page 97].

Lemma 15. *Let $\psi : S \rightarrow T$ be a morphism in $\mathcal{C}(\mathcal{A})$ with kernel (θ, K) . Then $(\theta_C, K(C))$ is the kernel of $\psi_C : S(C) \rightarrow T(C)$.*

Proposition 16. *The morphism α is a quasi-isomorphism in $Kom(\mathcal{C}(\mathcal{A}))$ if, and only if, α_C is a quasi-isomorphism in $Kom(\mathcal{A})$.*

Proof. Given $\alpha \in Hom_{Kom(\mathcal{C}(\mathcal{A}))}(F^\bullet, G^\bullet)$, by the isomorphism K , we can consider $\alpha : F \rightarrow G$ as a morphism in $\mathcal{C}(Kom(\mathcal{A}))$. We then have, for each object C in \mathcal{C} , a morphism $\alpha_C : F(C) \rightarrow G(C)$ in $Kom(\mathcal{A})$.

The key point in this proof consists in showing that $(H^i(\alpha))_C = H^i(\alpha_C)$.

Naturally, α_C is a quasi-isomorphism if, and only if, $H^i(\alpha_C)$ is an isomorphism. If the above equality is true, then $(H^i(\alpha))_C$ is also an isomorphism. But, that is true if, and only if, $H^i(\alpha)$ is an isomorphism, and if, and only if, α is a quasi-isomorphism.

Let us thus establish the desired equality. By definition we have that $H^{n+1}(F^\bullet) = Coker(a^n)$ and $H^{n+1}(G^\bullet) = Coker(b^n)$, where a^n and b^n are given by the following diagrams:

$$\begin{array}{ccc}
F^n & \xrightarrow{d_F^n} & F^{n+1} \xrightarrow{d_F^{n+1}} \dots \\
& \searrow a^n & \uparrow \\
& & Ker(d_F^{n+1})
\end{array}
\qquad
\begin{array}{ccc}
G^n & \xrightarrow{d_G^n} & G^{n+1} \xrightarrow{d_G^{n+1}} \dots \\
& \searrow b^n & \uparrow \\
& & Ker(d_G^{n+1}).
\end{array}$$

It then follows from Lemma 15 that

$$(H^{n+1}(F^\bullet))(C) = (Coker(a^n))(C) = Coker(a_C^n) = H^{n+1}(F(C))$$

and analogously $(H^{n+1}(G^\bullet))(C) = H^{n+1}(G(C))$. Moreover, we have the following commutative diagram

$$\begin{array}{ccccccc}
& & & \alpha_C^n & & & \\
& & & \curvearrowright & & & \\
F^n(C) & \xrightarrow{(d_F^n)_C} & F^{n+1}(C) & \xrightarrow{\alpha_C^{n+1}} & G^{n+1}(C) & \xleftarrow{(d_G^n)_C} & G^n(C) \\
& \searrow a_C^n & \uparrow & & \uparrow & \swarrow b_C^n & \\
& & \text{Ker}((d_F^{n+1})_C) & \xrightarrow{\alpha_C^{n+1}} & \text{Ker}((d_G^{n+1})_C) & & \\
& & \downarrow p_C & & \downarrow q_C & & \\
& & H^{n+1}(F(C)) & \xrightarrow{r_C} & H^{n+1}(G(C)) & &
\end{array}$$

where $q_C \circ \alpha_C^{n+1} \circ a_C^n = q_C \circ b_C^n \circ \alpha_C^n$, but $q_C \circ b_C^n = 0$ then $q_C \circ \alpha_C^{n+1} \circ a_C^n = 0$ and by the definition of cokernel of a_C^n , there is unique morphism r_C such that $r_C \circ p_C = q_C \circ \alpha_C^{n+1}$. However, both $(H^{n+1}(\alpha))_C$ and $H^{n+1}(\alpha_C)$ make the lower square commutative then, by uniqueness, $(H^{n+1}(\alpha))_C = H^{n+1}(\alpha_C)$. \square

3.3. The category $\mathcal{C}(D(\mathcal{A}))$. The objects of this category are functors between \mathcal{C} and $D(\mathcal{A})$. If F is an object of $\mathcal{C}(D(\mathcal{A}))$ and $C \xrightarrow{t} D$ is morphism in \mathcal{C} we have that $F(t) = [F_D(t)/F_C(t)]$ is a class of diagrams:

$$\begin{array}{ccc}
& F_{CD} & \\
F_C(t) \swarrow & & \searrow F_D(t) \\
F(C) & & F(D).
\end{array}$$

If $F \xrightarrow{\eta} G$ is a morphism in $\mathcal{C}(D(\mathcal{A}))$, for each $C \in \text{Ob}(\mathcal{C})$, we have that the morphism $F(C) \xrightarrow{\eta_C} G(C)$ in $D(\mathcal{A})$, where $\eta_C = [\eta_G/\eta_F]$ represents a class of diagrams:

$$\begin{array}{ccc}
& H_{\eta_C} & \\
\eta_F \swarrow & & \searrow \eta_G \\
F(C) & & G(C).
\end{array}$$

Since η is a natural transformation between F and G , for any given morphism $C \xrightarrow{t} D$ in \mathcal{C} the following diagram is commutative in $D(\mathcal{A})$:

$$\begin{array}{ccc}
F(C) & \xrightarrow{\eta_C} & G(C) \\
F(t) \downarrow & & \downarrow G(t) \\
F(D) & \xrightarrow{\eta_D} & G(D).
\end{array}$$

3.4. The functor T . Let $Q : \text{Kom}(\mathcal{A}) \rightarrow D(\mathcal{A})$ be the localization functor, that is, the functor which identifies the objects of the two categories $\text{Kom}(\mathcal{A})$ and $D(\mathcal{A})$, and associates, to a given a morphism $f : A^\bullet \rightarrow B^\bullet$ in $\text{Kom}(\mathcal{A})$, the class $[f/Id]$

represented by the roof:

$$(12) \quad \begin{array}{ccc} & A^\bullet & \\ Id \swarrow & & \searrow f \\ A^\bullet & & B^\bullet \end{array}$$

We can then define the induced functor $Q_{\mathcal{C}}$:

$$\begin{array}{ccc} Q_{\mathcal{C}} : \mathcal{C}(Kom(\mathcal{A})) & \longrightarrow & \mathcal{C}(D(\mathcal{A})) \\ F & \longmapsto & Q \circ F \\ \downarrow \eta & & \downarrow Q_{\mathcal{C}}(\eta) \\ G & & Q \circ G \end{array}$$

where $Q_{\mathcal{C}}(\eta) = \{Q(\eta_C) \in Hom_{D(\mathcal{A})}(QF(C), QG(C)); C \in Ob(\mathcal{C})\}$. Remember that $QF(C) = F(C)$, $QG(C) = G(C)$, and $Q(\eta_C) = [\eta_C/Id]$ which can be represented by the following diagram in $D(\mathcal{A})$.

$$(13) \quad \begin{array}{ccc} & F(C) & \\ Id \swarrow & & \searrow \eta_C \\ F(C) & & G(C) \end{array}$$

Proposition 17. *Let $K : Kom(\mathcal{C}(\mathcal{A})) \rightarrow \mathcal{C}(Kom(\mathcal{A}))$ be the equivalence of the Lema 13 and let $Q_{\mathcal{C}}$ be the induced functor defined above. Then the composition $Q_{\mathcal{C}} \circ K$ maps quasi-isomorphisms into isomorphisms.*

Proof. Take a quasi-isomorphism $\eta^\bullet \in Hom_{Kom(\mathcal{C}(\mathcal{A}))}((F^\bullet, d_F), (G^\bullet, d_G))$, that is $H^i(\eta^\bullet) : H^i(F^\bullet) \rightarrow H^i(G^\bullet)$ is an isomorphism in $\mathcal{C}(\mathcal{A})$ for all $i \in \mathbb{Z}$ therefore $(H^i(\eta^\bullet))_C : (H^i(F^\bullet))(C) \rightarrow (H^i(G^\bullet))(C)$ is an isomorphism in \mathcal{A} for each $C \in Ob(\mathcal{C})$ and for all $i \in \mathbb{Z}$.

We need to prove that $Q_{\mathcal{C}}(\eta)$ is an isomorphism in $\mathcal{C}(D(\mathcal{A}))$, that is, for each $C \in Ob(\mathcal{C})$, the morphism η_C in diagram (13) is a quasi-isomorphism in $Kom(\mathcal{A})$. But, by Proposition 16, $H^i(\eta_C) = (H^i(\eta^\bullet))_C$ then η_C is a quasi-isomorphisms. \square

Let $\overline{Q} : Kom(\mathcal{C}(\mathcal{A})) \rightarrow D(\mathcal{C}(\mathcal{A}))$ be the localization for the category $\mathcal{C}(\mathcal{A})$. Since $Q_{\mathcal{C}} \circ K : Kom(\mathcal{C}(\mathcal{A})) \rightarrow \mathcal{C}(D(\mathcal{A}))$ maps quasi-isomorphisms into isomorphisms, it follows, by definition of derived category, that there exists an unique functor $T : D(\mathcal{C}(\mathcal{A})) \rightarrow \mathcal{C}(D(\mathcal{A}))$ such that $T \circ \overline{Q} = Q_{\mathcal{C}} \circ K$, that is, the following diagram is commutative.

$$(14) \quad \begin{array}{ccc} Kom(\mathcal{C}(\mathcal{A})) & \xrightarrow{\overline{Q}} & D(\mathcal{C}(\mathcal{A})) \\ K \downarrow & & \downarrow T \\ \mathcal{C}(Kom(\mathcal{A})) & \xrightarrow{Q_{\mathcal{C}}} & \mathcal{C}(D(\mathcal{A})) \end{array}$$

The functor T can be described as follows:

$$(15) \quad \begin{array}{ccc} T : & D(\mathcal{C}(\mathcal{A})) & \longrightarrow \mathcal{C}(D(\mathcal{A})) \\ & \begin{array}{ccc} & H^\bullet & \\ \alpha \swarrow & & \searrow f \\ F^\bullet & & G^\bullet \end{array} & \longmapsto \begin{array}{c} T(F^\bullet) = Q_{\mathcal{C}}(F) \\ \downarrow T([f/\alpha]) \\ T(G^\bullet) = Q_{\mathcal{C}}(G), \end{array} \end{array}$$

where F and G are, respectively, the image of F^\bullet e G^\bullet by the functor K and $T([f/\alpha]) = \{(T([f/\alpha]))_C := [f_C/\alpha_C]; C \in \text{Ob}(\mathcal{C})\}$ is a natural transformation between $Q_{\mathcal{C}}(F)$ and $Q_{\mathcal{C}}(G)$.

It is not difficult to see that (15) fits into the commutative diagram (14); we must now argue that it does define a functor from $D(\mathcal{C}(\mathcal{A}))$ to $\mathcal{C}(D(\mathcal{A}))$.

First we see that T establishes indeed a relationship between $D(\mathcal{C}(\mathcal{A}))$ and $\mathcal{C}(D(\mathcal{A}))$.

Let $F^\bullet \in \text{Ob}(D(\mathcal{C}(\mathcal{A})))$ then $T(F^\bullet) = Q_{\mathcal{C}}(F)$, where $F = K(F^\bullet) \in \text{Ob}(\mathcal{C}(Kom(\mathcal{A})))$ and therefore $T(F^\bullet)$ is an object of $\mathcal{C}(D(\mathcal{A}))$. Let's see that $T([f/\alpha]) \in \text{Mor}(\mathcal{C}(D(\mathcal{A})))$, namely, let's see that $T([f/\alpha])$ is a natural transformation. For this, we need to check that given a morphism $C \xrightarrow{t} D$ in \mathcal{C} the diagram

$$\begin{array}{ccc} F(C) & \xrightarrow{[f_C/\alpha_C]} & G(C) \\ [F(t)/Id] \downarrow & & \downarrow [G(t)/Id] \\ F(D) & \xrightarrow{[f_D/\alpha_D]} & G(D) \end{array}$$

is commutative, that is, $[G(t)/Id] \circ [f_C/\alpha_C] = [f_D/\alpha_D] \circ [F(t)/Id]$.

Each side of equality is represented by below diagrams

$$\begin{array}{ccccc} & & H(C) & & \\ & Id \swarrow & & \searrow f_C & \\ H(C) & & & & G(C) \\ \alpha_C \swarrow & & f_C \searrow & Id \swarrow & \searrow G(t) \\ F(C) & & G(C) & & G(D) \end{array}$$

then $[G(t)/Id] \circ [f_C/\alpha_C] = [G(f) \circ f_C/\alpha_C]$. On the other hand, we have

$$\begin{array}{ccccc} & & H(C) & & \\ & \alpha_C \swarrow & & \searrow H(t) & \\ F(C) & & [1] & & H(D) \\ Id \swarrow & & F(t) \searrow & \alpha_D \swarrow & \searrow f_D \\ F(C) & & F(D) & & G(D) \end{array}$$

then $[f_D/\alpha_D] \circ [F(t)/Id] = [f_D \circ H(t)/\alpha_C]$.

The commutativity of the square [1] and the equality $[G(t) \circ f_C / \alpha_C] = [f_D \circ H(t) / \alpha_C]$ are true because the morphism $F^\bullet \xrightarrow{[f/\alpha]} G^\bullet$ in $D(\mathcal{C}(\mathcal{A}))$ induce, for each morphism $C \xrightarrow{t} D$ in \mathcal{C} , a diagram in $K(\mathcal{A})$ like (11).

To show that T is well defined we need to prove that if $[f_1/\alpha_1] = [f_2/\alpha_2]$ then $T([f_1/\alpha_1]) = T([f_2/\alpha_2])$.

Let $[f_1/\alpha_1] = [f_2/\alpha_2]$ then the following classes of diagrams are equivalent:

$$\begin{array}{ccc} & H_1 & \\ \alpha_1 \swarrow & & \searrow f_1 \\ F_1 & & G_1 \end{array} \qquad \begin{array}{ccc} & H_2 & \\ \alpha_2 \swarrow & & \searrow f_2 \\ F_2 & & G_2 \end{array}$$

Therefore exist quasi-isomorphisms $H_1^\bullet \xleftarrow{\gamma} R \xrightarrow{\delta} H_2^\bullet$ such that the following diagram is commutative:

$$\begin{array}{ccccc} & & R^\bullet & & \\ & \gamma \swarrow & & \searrow \delta & \\ & H_1^\bullet & & H_2^\bullet & \\ \alpha_1 \swarrow & & & & \searrow f_2 \\ F^\bullet & & \alpha_2 \swarrow & \searrow f_1 & G^\bullet \end{array}$$

As $T([f_1/\alpha_1]) = T([f_2/\alpha_2])$ if, and only if, $[(f_1)_C / (\alpha_1)_C] = [(f_2)_C / (\alpha_2)_C]$ for each $C \in \text{Ob}(\mathcal{C})$, we need to find quasi-isomorphisms $H_1(C) \xleftarrow{\gamma_C} R_C \xrightarrow{\delta_C} H_2(C)$ such that

$$\begin{array}{ccccc} & & R_C & & \\ & \gamma_C \swarrow & & \searrow \delta_C & \\ & H_1(C) & & H_2(C) & \\ (\alpha_1)_C \swarrow & & & & \searrow (f_2)_C \\ F(C) & & (\alpha_2)_C \swarrow & \searrow (f_1)_C & G(C) \end{array}$$

is commutative for each $C \in \text{Ob}(\mathcal{C})$. For this we use the quasi-isomorphisms

$$H_1^\bullet \xleftarrow{\gamma} R \xrightarrow{\delta} H_2^\bullet.$$

In order to prove that (14) is indeed a functor we need the following Lemma, whose proof is in [3, p. 253].

Lemma 18. *Given the diagram*

$$\begin{array}{ccc}
 & X'' & \xrightarrow{u''} Y'' \\
 t \swarrow & & \searrow s \\
 X & \xrightarrow{f} Y & \\
 \downarrow & & \downarrow g \\
 X' & \xrightarrow{u'} Y' &
 \end{array}$$

in $K(\mathcal{A})$, making a change, if necessary, the roofs representing $X \xleftarrow{t} X'' \xrightarrow{f} X'$ and $Y \xleftarrow{s} Y'' \xrightarrow{g} Y'$ in $D(\mathcal{A})$, we can find a morphism $X'' \xrightarrow{u''} Y''$ such that the two squares are commutative in $K(\mathcal{A})$, that is, $s \circ u'' = u \circ t$ and $g \circ u'' = u' \circ f$.

Let us finally prove that T satisfies both functor properties. First, let $F^\bullet \xrightarrow{[f/\alpha]} G^\bullet$ and $G^\bullet \xrightarrow{[g/\beta]} E^\bullet$ morphisms in $D(\mathcal{C}(\mathcal{A}))$; it is necessary to check that $T([f/\alpha] \circ [g/\beta]) = T([f/\alpha]) \circ T([g/\beta])$.

By the definition of derived category, there are β' and f' such that

$$T([f/\alpha] \circ [g/\beta]) = T([g \circ f' / \alpha \circ \beta']) := \{[g_C \circ f'_C / \alpha_C \circ \beta'_C]; C \in \text{Ob}(\mathcal{C})\}.$$

On the other hand, for each $C \in \text{Ob}(\mathcal{C})$, there are β''_C and f''_C such that

$$T([f/\alpha]) \circ T([g/\beta]) = \{[g_C \circ f''_C / \alpha_C \circ \beta''_C]; C \in \text{Ob}(\mathcal{C})\}.$$

By Lemma 18, there is ϕ for which the squares of the following diagram

$$\begin{array}{ccc}
 W_C & \xrightarrow{\phi} & V(C) \\
 \alpha_C \circ \beta''_C \swarrow & & \searrow \alpha_C \circ \beta'_C \\
 F(C) & \xrightarrow{g_C \circ f''_C} F(C) & \\
 \downarrow & & \downarrow g_C \circ f'_C \\
 E(C) & \xrightarrow{Id} & E(C)
 \end{array}$$

are commutative. It is important to note that, in this case, ϕ is a quasi-isomorphism. Then we have that

$$\begin{array}{ccccc}
 & & W_C & & \\
 & Id \swarrow & & \searrow \phi & \\
 & W_C & & V(C) & \\
 \alpha_C \circ \beta''_C \swarrow & & \searrow \alpha_C \circ \beta'_C & & \searrow g_C \circ f'_C \\
 F(C) & & & & E(C)
 \end{array}$$

is commutative so $[g_C \circ f''_C / \alpha_C \circ \beta''_C] = [g_C \circ f'_C / \alpha_C \circ \beta'_C]$. Therefore $T([f/\alpha] \circ [g/\beta]) = T([f/\alpha]) \circ T([g/\beta])$.

The equality $T(Id_{F^\bullet}) = Id_{T(F^\bullet)}$ follows directly of definition of T . This completes the argument showing that (14) does define a functor. One can also prove the existence of the functor T for the categories $D^*(\mathcal{C}(\mathcal{A}))$ and $\mathcal{C}(D^*(\mathcal{A}))$, where $*$ = $b, -, +$.

In general, T is not an equivalence of categories. However, under certain conditions we can show that T is full and faithful. In order to set up these conditions, let us recall the notion of quiver.

Definition 19. A quiver $Q = (Q_0, Q_1, t, h)$ is an oriented graph, i.e., it consists of two sets

- Q_0 , the set of vertices;
- Q_1 , the set of arrows between vertices;

plus two maps t and h between them:

$$\begin{aligned} t : Q_1 &\longrightarrow Q_0 \\ a &\longmapsto t(a) = \text{initial vertex;} \\ h : Q_1 &\longrightarrow Q_0 \\ a &\longmapsto h(a) = \text{end vertex.} \end{aligned}$$

A path in the quiver Q is a sequence of arrows $p = a_1 a_2 \dots a_n$ with $h(a_{i+1}) = t(a_i)$ for $1 \leq i < n$. We define $t(p) = t(a_n)$ and $h(p) = h(a_1)$. We call the paths e_i trivial paths and define $h(e_i) = t(e_i) = i$, for all $i \in Q_0$.

For each quiver Q it is possible to associate a category \mathcal{Q} , where each vertex i is seen as an object, and each path p is seen as a morphism in $\text{Hom}_{\mathcal{Q}}(t(p), h(p))$. We say that the category \mathcal{Q} is generated by the quiver Q .

Remark 20. Alternatively, the category $\mathcal{Q}(\mathcal{A})$ can also be described as the category of representations of the quiver Q into the category \mathcal{A} ; such category is often denoted by $\text{Rep}(Q, \mathcal{A})$. Its objects (called representations) consist of

- a family $\{A_i | i \in Q_0\}$, with $A_i \in \text{Ob}(\mathcal{A})$, and
- a family $\{\phi_a : A_{t(a)} \longrightarrow A_{h(a)} | a \in Q_1\}$.

Given two representations (A, ϕ) and (B, φ) a morphism $f : (A, \phi) \longrightarrow (B, \varphi)$ in $\text{Rep}(Q, \mathcal{A})$ is a family of morphism $f_i \in \text{Hom}_{\mathcal{A}}(A_i, B_i)$, $i \in Q_0$, such that for each arrow $a \in Q_1$, $\varphi_a \circ f_{t(a)} = f_{h(a)} \circ \phi_a$.

When \mathcal{A} is the category of (finite dimensional) vector spaces over a field, $\mathcal{Q}(\mathcal{A})$ is precisely the category of (finite dimensional) modules over the path algebra of Q .

We are finally in position to establish the first key result of this paper.

Theorem 21. Let Q be a quiver, and let \mathcal{Q} be the category it generates. Then the functor $T : D(\mathcal{Q}(\mathcal{A})) \longrightarrow \mathcal{Q}(D(\mathcal{A}))$ is full and faithful.

Proof. We must to show that given F^\bullet and G^\bullet objects in $D(\mathcal{Q}(\mathcal{A}))$, the map $T : \text{Hom}_{D(\mathcal{Q}(\mathcal{A}))}(F^\bullet, G^\bullet) \longrightarrow \text{Hom}_{\mathcal{Q}(D(\mathcal{A}))}(T(F^\bullet), T(G^\bullet))$ is a bijective map. Remember that $T(F^\bullet) = Q_{\mathcal{Q}}(K(F^\bullet)) = Q \circ F$, and that Q denotes the localization functor $Q : \text{Kom}(\mathcal{A}) \longrightarrow D(\mathcal{A})$.

Set $[f/\alpha]$ and $[g/\beta]$ in $\text{Hom}_{D(\mathcal{Q}(\mathcal{A}))}(F^\bullet, G^\bullet)$ given by:

$$\begin{array}{ccc} & E^\bullet & \\ \alpha \swarrow & & \searrow f \\ F^\bullet & & G^\bullet \end{array} \qquad \begin{array}{ccc} & D^\bullet & \\ \beta \swarrow & & \searrow g \\ F^\bullet & & G^\bullet \end{array}$$

respectively, and such that $T([f/\alpha]) = T([g/\beta])$. In other words,

$$\{[f_i/\alpha_i] : i \in \text{Ob}(\mathcal{Q})\} = \{[g_i/\beta_i] : i \in \text{Ob}(\mathcal{Q})\}.$$

Thus, for each $i \in \text{Ob}(\mathcal{Q})$, $[f_i/\alpha_i] = [g_i/\beta_i]$, i.e. there are quasi-isomorphisms

$$E(i)^\bullet \xleftarrow{\gamma_i} W(i)^\bullet \xrightarrow{\delta_i} D(i)^\bullet$$

such that the following diagram is commutative:

$$\begin{array}{ccccc} & & W(i)^\bullet & & \\ & \swarrow \gamma_i & & \searrow \delta_i & \\ E(i)^\bullet & & & & D(i)^\bullet \\ \swarrow \alpha_i & & \searrow \beta_i & & \swarrow f_i \\ F(i)^\bullet & & & & G(i)^\bullet \end{array}$$

Let $i \xrightarrow{p} j$ be a morphism in \mathcal{Q} . By Lemma 18, there is $W(p)$ such that the two squares of the diagram

$$(16) \quad \begin{array}{ccccc} & & W(i)^\bullet & \xrightarrow{W(p)} & W(j)^\bullet \\ & \swarrow \gamma_i & & & \swarrow \gamma_j \\ E(i)^\bullet & \xrightarrow{\delta_i} & E(j)^\bullet & & \\ \swarrow \alpha_i & & \searrow \beta_j & & \swarrow f_j \\ D(i)^\bullet & \xrightarrow{D(p)} & D(j)^\bullet & & \end{array}$$

are commutative. Hence we can define a functor

$$\begin{array}{ccc} W : & \mathcal{Q} & \longrightarrow K(\mathcal{A}) \\ & i & \longmapsto W(i)^\bullet \\ & \downarrow p & \\ & j & \longmapsto W(j)^\bullet \end{array}$$

In order to have a well-defined functor, we set that if $i \xrightarrow{p} j$ and $j \xrightarrow{q} k$ are morphisms in \mathcal{Q} , then we define $W(p \circ q) = W(p) \circ W(q)$.

From W we can generate a functor $W' : \mathcal{Q} \longrightarrow \text{Kom}(\mathcal{A})$ choosing a representative of the class $W(p)$ in $\text{Kom}(\mathcal{A})$, which we denote by W'_p . Again, we have that W' is well-defined functor. Then:

$$\begin{array}{ccc} W' : & \mathcal{Q} & \longrightarrow \text{Kom}(\mathcal{A}) \\ & i & \longmapsto W'(i)^\bullet \\ & \downarrow p & \\ & j & \longmapsto W'(j)^\bullet \end{array}$$

where $W'(i)^\bullet = W(i)^\bullet$ for each i object in \mathcal{Q} .

Furthermore, by the commutativity of diagram (16), we have that

$$\gamma = \{\gamma_i \in \text{Hom}(W'(i)^\bullet, E(i)^\bullet); i \in \text{Ob}(\mathcal{Q})\} \quad \text{and}$$

$$\delta = \{\delta_i \in \text{Hom}(W(i)'^\bullet, D(i)^\bullet); i \in \text{Ob}(\mathcal{Q})\}$$

are natural transformations from W' into E , and from W' into D , respectively. Using the equivalence K , we have in $D(\mathcal{C}(\mathcal{A}))$ the following diagram:

$$\begin{array}{ccccc} & & W'^\bullet & & \\ & \swarrow \gamma & & \searrow \delta & \\ & E^\bullet & & D^\bullet & \\ \alpha \swarrow & & & & \searrow g \\ F^\bullet & & & & G^\bullet \\ & \nwarrow \beta & & \nearrow f & \end{array}$$

where γ and δ are quasi-isomorphisms, as consequence of γ_i and δ_i are quasi-isomorphisms for each $i \in \text{Ob}(\mathcal{Q})$. Therefore $[f/\alpha] = [g/\beta]$, and

$$T : \text{Hom}_{D(\mathcal{Q}(\mathcal{A}))}(F^\bullet, G^\bullet) \longrightarrow \text{Hom}_{\mathcal{Q}(D(\mathcal{A}))}(T(F^\bullet), T(G^\bullet))$$

is an injective map. This completes the proof that T is faithful.

To see that T is also full, let $f \in \text{Hom}_{\mathcal{Q}(D(\mathcal{A}))}(Q \circ F, Q \circ G)$; we must define a morphism $[h/\phi] \in \text{Hom}_{D(\mathcal{Q}(\mathcal{A}))}(F^\bullet, G^\bullet)$ such that $T([h/\phi]) = f$. Now, f is a natural transformation between $Q \circ F : \mathcal{Q} \rightarrow D(\mathcal{A})$ and $Q \circ G : \mathcal{Q} \rightarrow D(\mathcal{A})$. We can define $f = \{[f_i/\alpha_i]; i \in \text{Ob}(\mathcal{Q})\}$ such that, given $i \xrightarrow{p} j$, we have the commutative diagram in $D(\mathcal{A})$:

$$(17) \quad \begin{array}{ccc} F(i) & \xrightarrow{[f_i/\alpha_i]} & G(i) \\ [F(p)/Id] \downarrow & & \downarrow [G(p)/Id] \\ F(j) & \xrightarrow{[f_j/\alpha_j]} & G(j), \end{array}$$

remembering that $Q \circ F(i) = F(i)$ for all $i \in \text{Ob}(\mathcal{Q})$, and $Q \circ F(p) = [F(p)/Id]$ for all $p \in \text{Mor}(\mathcal{Q})$. Similarly for $Q \circ G$.

On the other hand, $[h/\phi]$ can be represented by the roof

$$\begin{array}{ccc} & H^\bullet & \\ \phi \swarrow & & \searrow h \\ F^\bullet & & G^\bullet \end{array}$$

in $D(\mathcal{Q}(\mathcal{A}))$, and for each morphism $i \xrightarrow{p} j$ in \mathcal{Q} , we have:

$$(18) \quad \begin{array}{ccc} H(i)^\bullet & \xrightarrow{H(p)} & H(j)^\bullet \\ \phi_i \swarrow & & \searrow \alpha_j \\ F(i)^\bullet & \xrightarrow[h_i]{F(p)} & F(j)^\bullet \\ \swarrow & & \searrow h_j \\ G(i)^\bullet & \xrightarrow{G(p)} & G(j)^\bullet \end{array}$$

Thus take $h_i = f_i$, $\phi_i = \alpha_i$ and $H(i) = H_i$ for all $i \in \text{Ob}(\mathcal{Q})$. In order to define H^\bullet , we still need define who is $H(p)$. However, by Lemma (18) guarantees the existence of such $H(p)$. And again, we can take $H : \mathcal{Q} \rightarrow \text{Kom}(\mathcal{A})$. So $[h/\phi]$ thus defined satisfies $T([h/\phi]) = f$. \square

We conclude this section with an example that shows that T is not in general an equivalence between $D(\mathcal{Q}(\mathcal{A}))$ and $\mathcal{Q}(D(\mathcal{A}))$. Indeed, Let H be the cohomology functor $H : \text{Kom}(\mathcal{A}) \rightarrow \text{Kom}_0(\mathcal{A})$, $H((A^n, d^n)) = (H^i(A^\bullet), 0)$, $H(f : A^\bullet \rightarrow B^\bullet) = (H^i(f))$. Since H maps quasi-isomorphisms in isomorphism, it factors through $D(\mathcal{A})$, that is, we have a functor $R : \text{Kom}_0(\mathcal{A}) \rightarrow D(\mathcal{A})$ such that $Q = R \circ H$, where Q is the localization.

Before proceeding, let us recall the following fact (cf. [3, III.4, page 146]).

Proposition 22. *Let \mathcal{A} be an abelian category. \mathcal{A} is semisimple if, and only if, the functor R is an equivalence of categories.*

Let \mathcal{A} be a semisimple category. by the Proposition 22, the functor $R : \text{Kom}_0(\mathcal{A}) \rightarrow D(\mathcal{A})$ is an equivalence, then, using the Proposition 11, the induced functor $R_C : \mathcal{C}(\text{Kom}_0(\mathcal{A})) \rightarrow \mathcal{C}(D(\mathcal{A}))$ is also an equivalence. It then follows from Corollary 14 that if \mathcal{A} is semisimple, then $\mathcal{C}(D(\mathcal{A}))$ and $\text{Kom}_0(\mathcal{C}(\mathcal{A}))$ are equivalent. On the other hand, it is not difficult to see that $\mathcal{C}(\mathcal{A})$ may not be semisimple, and thus, by Proposition 22 above, $D(\mathcal{C}(\mathcal{A}))$ and $\text{Kom}_0(\mathcal{C}(\mathcal{A}))$ are not equivalent therefore $D(\mathcal{C}(\mathcal{A}))$ and $\mathcal{C}(D(\mathcal{A}))$ are also not equivalent.

For example, consider, let \mathcal{Q} be the category induced by the quiver $\bullet \rightarrow \bullet$, i.e. \mathcal{Q} has two objects $\text{Ob}(\mathcal{Q}) = \{Q_1, Q_2\}$ and three morphisms $\text{Mor}(\mathcal{Q}) = \{Id_{Q_1}, Id_{Q_2}, a : Q_1 \rightarrow Q_2\}$. On the other hand, let \mathcal{V} be the category of finite dimensional vector spaces over a field \mathbb{C} . Clearly, \mathcal{V} is semisimple, but $\mathcal{Q}(\mathcal{V})$ is not (in fact, $\mathcal{Q}(\mathcal{V})$ is just the category of modules over the path algebra $\mathbb{C}Q$). It follows that $D(\mathcal{Q}(\mathcal{V}))$ can be regarded as a proper full subcategory of $\mathcal{Q}(D(\mathcal{V}))$.

4. COMPARISON BETWEEN $R(F_C)$ AND $(RF)_C$

Set \mathcal{C} a small category, \mathcal{A} and \mathcal{B} abelian categories; assume that \mathcal{A} has enough injectives. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive, left exact functor. By Propositions 9 and 10, we know that the induced functor $F_C : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{B})$ is also additive and left exact. Moreover, if $\mathcal{C}(\mathcal{A})$ has enough injectives, the induced functor F_C admits the extension $R(F_C) : D^+(\mathcal{C}(\mathcal{A})) \rightarrow D^+(\mathcal{C}(\mathcal{B}))$, its right derived functor.

On the other hand, starting from the same functor $F : \mathcal{A} \rightarrow \mathcal{B}$, one may first consider its right derived extension $RF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$, and then define the induced functor $(RF)_C : \mathcal{C}(D^+(\mathcal{A})) \rightarrow \mathcal{C}(D^+(\mathcal{B}))$.

We will now study the relationship between these two functors, $R(F_{\mathcal{C}})$ and $(RF)_{\mathcal{C}}$. In order to do this, we must first set up some notation. We use $Q_{\mathcal{A}}$, $K_{\mathcal{A}}$ and $T_{\mathcal{A}}$ for the functors Q , K and T defined in the previous Section relatively to the category \mathcal{A} ; similarly, we use $Q_{\mathcal{B}}$, $K_{\mathcal{B}}$ and $T_{\mathcal{B}}$ for the same functors relatively to the category \mathcal{B} .

Let $\mathcal{C}(\mathcal{A})$ be a category with enough injectives; for instance, referring either to Proposition 4 or to Corollary 5, assume that either \mathcal{C} has a finite number of objects and morphisms, or \mathcal{A} is a complete category.

Being A^{\bullet} an object in $D(\mathcal{C}(\mathcal{A}))$, and since $\mathcal{C}(\mathcal{A})$ has enough injectives, there is a quasi-isomorphism $\alpha : A^{\bullet} \rightarrow I(A^{\bullet})^{\bullet}$, where $I(A^{\bullet})^{\bullet}$ is a complex of injective objects. The existence of this quasi-isomorphism is established in [2, Section A.4.5].

We have therefore an isomorphism

$$A^{\bullet} \xleftarrow{Id} A^{\bullet} \xrightarrow{\alpha} I(A^{\bullet})^{\bullet}$$

in $D(\mathcal{C}(\mathcal{A}))$. By the commutativity of the diagram (14), $T_{\mathcal{A}}([\alpha/Id])$ is a natural transformation in $\mathcal{C}(D(\mathcal{A}))$, between $Q_{\mathcal{A}} \circ A$ and $Q_{\mathcal{A}} \circ I(A^{\bullet})$, where $A = K_{\mathcal{A}}(A^{\bullet})$ and $I(A^{\bullet}) = K_{\mathcal{A}}(I(A^{\bullet})^{\bullet})$, that is,

$$T_{\mathcal{A}}([\alpha/Id]) = \{[\alpha_C/Id] \in Hom_{D(\mathcal{A})}(A(C), I(A^{\bullet})(C)) : C \in Ob(\mathcal{C})\}$$

such that the following diagram is commutative for each morphism $C \xrightarrow{t} D$ in \mathcal{C} :

$$\begin{array}{ccc} A(C) & \xrightarrow{[\alpha_C/Id]} & I(A^{\bullet})(C) \\ [A(t)/Id] \downarrow & & \downarrow [I(A^{\bullet})(t)/Id] \\ A(D) & \xrightarrow{[\alpha_D/Id]} & I(A^{\bullet})(D). \end{array}$$

Applying the functor $(RF)_{\mathcal{C}}$ to $T_{\mathcal{A}}([\alpha/Id])$, we obtain a natural transformation in $\mathcal{C}(D(\mathcal{B}))$, namely $(RF)_{\mathcal{C}}(T_{\mathcal{A}}([\alpha/Id])) : RF \circ Q_{\mathcal{A}} \circ A \rightarrow RF \circ Q_{\mathcal{A}} \circ I(A^{\bullet})$. So, if $C \xrightarrow{t} D$ is a morphism in \mathcal{C} we can define $RF \circ Q_{\mathcal{A}} \circ A(C \xrightarrow{t} D)$ through two steps. First,

$$\begin{array}{ccc} C & \xrightarrow{Q_{\mathcal{A}} \circ A} & A(C) \\ t \downarrow & & \swarrow Id \quad \searrow A(t) \\ D & & A(C) \quad A(D) \end{array}.$$

Secondly, applying RF (cf. [5]):

$$\begin{array}{ccc} & A(C) & \\ q_A(C) \swarrow & & \searrow q_A(D) \circ A(t) \\ I(A(C))^{\bullet} & \xrightarrow{I(A(t))} & I(A(D))^{\bullet} \end{array},$$

where $q_A(C)$ e $q_A(D)$ are quasi-isomorphism whose existence is guaranteed by the fact that \mathcal{A} has enough injectives. Finally, $RF \circ Q_{\mathcal{A}} \circ A(C \xrightarrow{t} D)$ is:

$$\begin{array}{ccc}
 & KF(I(A(C))^\bullet) & \\
 Id \swarrow & & \searrow KF(I(A(t))) \\
 KF(I(A(C))^\bullet) & & KF(I(A(D))^\bullet).
 \end{array}$$

Similarly, one defines $RF \circ Q_{\mathcal{A}} \circ I(A^\bullet)(C \xrightarrow{t} D)$:

$$\begin{array}{ccc}
 & KF(I(A^\bullet)(C)) & \\
 Id \swarrow & & \searrow KF(I(A^\bullet)(t)) \\
 KF(I(A^\bullet)(C)) & & KF(I(A^\bullet)(D)).
 \end{array}$$

Moreover

$$(RF)_C(T_{\mathcal{A}}([\alpha/Id])) = \{RF([\alpha_C/Id]) : C \in Ob(\mathcal{C})\} = \{[KF(I(\alpha_C))/Id] : C \in Ob(\mathcal{C})\},$$

namely, for each C , $I(\alpha_C)$ is determined by the diagram:

$$(19) \quad \begin{array}{ccc}
 & A(C) & \\
 Id \swarrow & & \searrow \alpha_C \\
 A(C) & & I(A^\bullet)(C) \\
 q_{A(C)} \downarrow & & \downarrow Id \\
 I(A(C))^\bullet & \xrightarrow{I(\alpha_C)} & I(A^\bullet)(C).
 \end{array}$$

Noting that, as $q_A(C)$ and α_C are quasi-isomorphisms, $I(\alpha_C)$ is also a quasi-isomorphism. Therefore $(RF)_C(T_{\mathcal{A}}([\alpha/Id])) = \{[KF(I(\alpha_C))/Id] : C \in Ob(\mathcal{C})\}$, such that, for each morphism $C \xrightarrow{t} D$, the diagram

$$(20) \quad \begin{array}{ccc}
 KF(I(A(C))^\bullet) & \xrightarrow{[KF(I(\alpha_C))/Id]} & KF(I(A^\bullet)(C)) \\
 \downarrow [KF(I(A(t)))/Id] & & \downarrow [KF(I(A^\bullet)(t))/Id] \\
 KF(I(A(D))^\bullet) & \xrightarrow{[KF(I(\alpha_D))/Id]} & KF(I(A^\bullet)(D))
 \end{array}$$

is commutative; in other words, we have the following commutative diagram in $K(\mathcal{B})$:

$$\begin{array}{ccccc}
 & & W^\bullet & & \\
 & \swarrow \gamma & & \searrow \gamma & \\
 & KF(I(A(C))^\bullet) & & KF(I(A(C))^\bullet) & \\
 & \swarrow Id & & \searrow (2) & \\
 KF(I(A(C))^\bullet) & & & & KF(I(A^\bullet)(D)), \\
 & \nwarrow Id & & \nearrow (1) &
 \end{array}$$

where $(1) = KF(I(A^\bullet)(t)) \circ KF(I(\alpha_C))$ and $(2) = KF(I(\alpha_D)) \circ KF(I(A(t)))$. Therefore,

$$(21) \quad \left(KF(I(\alpha_D)) \circ KF(I(A(t))) \right) \sim \left(KF(I(A^\bullet)(t)) \circ KF(I(\alpha_C)) \right)$$

It is worthy noting that $I(\alpha_C)$ is a quasi-isomorphism between complexes of injective objects. Under these conditions it is easy to verify that $I(\alpha_C)$ is an isomorphism in $K(\mathcal{A})$ and therefore $KF(I(\alpha_C))$ is an isomorphisms in $K(\mathcal{B})$, for all $C \in Ob(\mathcal{C})$.

Lemma 23. *Let \mathcal{A} be an abelian category with enough injectives, and let \mathcal{C} be a small category, such that, either \mathcal{A} is complete or \mathcal{C} has a finite number of objects and morphisms. Then, for every left exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$, the following diagram*

$$(22) \quad \begin{array}{ccc}
 D^+(\mathcal{C}(\mathcal{A})) & \xrightarrow{R(F_C)} & D^+(\mathcal{C}(\mathcal{B})) \\
 T_A \downarrow & & \downarrow T_B \\
 \mathcal{C}(D^+(\mathcal{A})) & \xrightarrow{(RF)_C} & \mathcal{C}(D^+(\mathcal{B}))
 \end{array}$$

is commutative.

Proof. The proof is done in two steps. In the first step, it is demonstrated that diagram (22) is commutative for the objects. In the second step, the same is done for the morphisms.

1st step: Set $A^\bullet \in Ob(D^+(\mathcal{C}(\mathcal{A})))$, so that $T_A(A^\bullet) = Q_A \circ A : \mathcal{C} \rightarrow D(\mathcal{A})$ is defined by:

$$\begin{array}{ccc}
 C & \mapsto & A(C) \\
 t \downarrow & & \swarrow Id \quad \searrow A(t) \\
 D & & A(C) \quad A(D),
 \end{array}$$

where $A = K_{\mathcal{A}}(A^\bullet) \in Ob(\mathcal{C}(Kom(\mathcal{A})))$.

The composition of $(RF)_C$ with $T_A(A^\bullet)$ is a functor between \mathcal{C} and $D(\mathcal{B})$ defined by

$$\begin{array}{ccc}
 C & \mapsto & KF(I(A(C))^\bullet) \\
 t \downarrow & & \swarrow Id \quad \searrow KF(I(A(t))) \\
 D & & KF(I(A(C))^\bullet) \quad (I) \quad KF(I(A(D))^\bullet).
 \end{array}$$

On the other hand,

$$R(F_{\mathcal{C}})(A^\bullet) = K(F_{\mathcal{C}})(I(A^\bullet)^\bullet) = (F_{\mathcal{C}}(I(A^\bullet)))^\bullet = (F \circ I(A^\bullet))^\bullet$$

belongs to $Ob(D(\mathcal{C}(\mathcal{B})))$. Thus, defining $F \circ I(A^\bullet) = K_{\mathcal{B}}((F \circ I(A^\bullet))^\bullet)$, we have

$$T_{\mathcal{B}} \circ R(F_{\mathcal{C}})(A^\bullet) = Q_{\mathcal{B}} \circ (F \circ I(A^\bullet)) : \mathcal{C} \longrightarrow D(\mathcal{B})$$

$$\begin{array}{ccc} C & \mapsto & (F \circ I(A^\bullet))(C) \\ t \downarrow & & \swarrow \text{Id} \quad \searrow (F \circ I(A^\bullet))(t) \\ D & & (F \circ I(A^\bullet))(C) \quad (F \circ I(A^\bullet))(D) \end{array}$$

Since $(F \circ I(A^\bullet))(C) = KF(I(A^\bullet)(C))$ and $(F \circ I(A^\bullet))(t) = KF(I(A^\bullet)(t))$, for all $C \in Ob(\mathcal{C})$ and for all morphism t of \mathcal{C} then $T_{\mathcal{B}} \circ R(F_{\mathcal{C}})(A^\bullet)$ can be defined by:

$$\begin{array}{ccc} C & \mapsto & KF(I(A^\bullet)(C)) \\ t \downarrow & & \swarrow \text{Id} \quad \searrow KF(I(A^\bullet)(t)) \\ D & & KF(I(A^\bullet)(C)) \quad (II) \quad KF(I(A^\bullet)(D)) \end{array}$$

Therefore, just need to check that the roofs (I) and (II) are equivalent.

Since $KF(I(\alpha_C))$ and $KF(I(\alpha_D))$ are isomorphisms, the roof (I) is equivalent to:

$$\begin{array}{ccc} & KF(I(A(C))) & \\ KF(I(\alpha_C)) \swarrow & & \searrow KF(I(A(t))) \circ KF(I(\alpha_D)) \\ KF(I(A^\bullet)(C)) & & KF(I(A^\bullet)(D)) \end{array}$$

Then, by (21), we have following commutative diagram in $K(\mathcal{B})$:

$$\begin{array}{ccccc} & & KF(I(A(C))) & & \\ & \swarrow \text{Id} & & \searrow KF(I(\alpha_C)) & \\ & KF(I(A(C))) & & & KF(I(A^\bullet)(C)) \\ & \downarrow KF(I(\alpha_C)) & \swarrow \text{Id} & \searrow KF(I(A(t))) \circ KF(I(A(D))) & \downarrow KF(I(A^\bullet)(t)) \\ & KF(I(A^\bullet)(C)) & & & KF(I(A^\bullet)(D)) \end{array}$$

It follows that (22) is commutative in the objects.

2nd step: Let $[f/\phi]$ be a morphism in $D(\mathcal{C}(\mathcal{A}))$ represented by the roof:

$$\begin{array}{ccc} & L^\bullet & \\ \phi \swarrow & & \searrow f \\ E^\bullet & & G^\bullet \end{array}$$

Then $T_{\mathcal{A}}([f/\phi]) = \{[f_C/\phi_C] : C \in \text{Ob}(\mathcal{C})\}$, and for any morphism $C \xrightarrow{t} D$ in \mathcal{C} , the diagram

$$\begin{array}{ccc} E(C) & \xrightarrow{[f_C/\phi_C]} & G(C) \\ [E(t)/Id] \downarrow & & \downarrow [G(t)/Id] \\ E(D) & \xrightarrow{[f_D/\phi_D]} & G(D) \end{array}$$

is commutative. Applying $(RF)_{\mathcal{C}}$

$$(RF)_{\mathcal{C}} \circ T_{\mathcal{A}}([f/\phi]) = \{RF([f_C/\phi_C]) : C \in \text{Ob}(\mathcal{C})\}.$$

This means that for each $C \in \text{Ob}(\mathcal{C})$ we can construct $I(f_C/\phi_C)$,

$$\begin{array}{ccc} & L(C) & \\ \phi_C \swarrow & & \searrow f_C \\ E(C) & & G(C) \\ q\text{-iso} \downarrow & & \downarrow q\text{-iso} \\ I(E(C)) & \xrightarrow{I(f_C/\phi_C)} & I(G(C)), \end{array}$$

and $(RF)_{\mathcal{C}} \circ T_{\mathcal{A}}([f/\phi]) = \{[KF(I(f_C/\phi_C))/Id] : C \in \text{Ob}(\mathcal{C})\}$, that is, for each C the class is represented by

$$\begin{array}{ccc} & KF(I(E(C))) & \\ Id \swarrow & & \searrow KF(I(f_C/\phi_C)) \\ KF(I(E(C))) & & KF(I(G(C))). \end{array}$$

On the other hand, applying $R(F_{\mathcal{C}})$ over $[f/\phi]$ we have

$$\begin{array}{ccc} & L^{\bullet} & \\ \phi \swarrow & & \searrow f \\ E^{\bullet} & & G^{\bullet} \\ q\text{-iso} \downarrow & & \downarrow q\text{-iso} \\ I(E^{\bullet}) & \xrightarrow{I(f/\phi)} & I(G^{\bullet}) \end{array}$$

$$\begin{array}{ccc} & K(F_{\mathcal{C}})(I(E^{\bullet})) & \\ Id \swarrow & & \searrow K(F_{\mathcal{C}})(I(f/\phi)) \\ K(F_{\mathcal{C}})(I(E^{\bullet})) & & K(F_{\mathcal{C}})(I(G^{\bullet})), \end{array}$$

Where $K(F_C)(I(E^\bullet)) = (F \circ I(E^\bullet))^\bullet$ and $K(F_C)(I(f/\phi)) = [F_C(I(f/\phi))]$ then the following roof is equivalent to the roof above in $D(\mathcal{C}(\mathcal{B}))$

$$\begin{array}{ccc} & (F \circ I(E^\bullet))^\bullet & \\ \swarrow Id & & \searrow [F_C(I(f/\phi))] \\ (F \circ I(E^\bullet))^\bullet & & (F \circ I(G^\bullet))^\bullet, \end{array}$$

Applying T_B :

$$T_B([F_C(I(f/\phi))/Id]) = \{[(F_C(I(f/\phi)))_C/Id] : C \in \text{Ob}(\mathcal{C})\}.$$

and, by definition F_C , $(F_C(I(f/\phi)))_C = (F((I(f/\phi))_C))^\bullet = KF((I(f/\phi))_C)$ thus,

$$T_B([F_C(I(f/\phi))/Id]) = \{[KF((I(f/\phi))_C)/Id] : C \in \text{Ob}(\mathcal{C})\}.$$

So, to have $(RF)_C \circ T_A([f/\phi]) = T_B \circ R(F)_C([f/\phi])$, we need to prove that

$$KF(I(f_C/\phi_C)) \sim KF((I(f/\phi))_C).$$

Let $\xi : E^\bullet \rightarrow I(E^\bullet)$ and $\theta : G^\bullet \rightarrow I(G^\bullet)$ be quasi-isomorphisms. We then have the following commutative diagrams, determined as in (19):

$$\begin{array}{ccc} & E(C) & \\ q_{E(C)} \swarrow & & \searrow \xi_C \\ I(E(C))^\bullet & \xrightarrow{I(\xi_C)} & I(E^\bullet)(C) \end{array} \quad \begin{array}{ccc} & G(C) & \\ q_{G(C)} \swarrow & & \searrow \theta_C \\ I(G(C))^\bullet & \xrightarrow{I(\theta_C)} & I(G^\bullet)(C), \end{array}$$

where $q_{E(C)}$, $q_{G(C)}$, ξ_C , θ_C are quasi-isomorphisms, $I(\xi_C)$ and $I(\theta_C)$ are isomorphisms. Then, by the definition of $I(f/\phi)$ and of $I(f_C/\phi_C)$, the diagram

$$\begin{array}{ccccc} & L(C) & & & \\ & \phi_C \swarrow & & \searrow f_C & \\ E(C) & & G(C) & & \\ \downarrow q_{E(C)} & & \downarrow q_{G(C)} & & \\ I(E(C)) & \xrightarrow{I(f_C/\phi_C)} & I(G(C)) & & \\ \downarrow I(\xi_C) & & \downarrow I(\theta_C) & & \\ I(E^\bullet)(C) & \xrightarrow{(I(f/\phi))_C} & I(G^\bullet)(C). & & \end{array}$$

is commutative in $K(\mathcal{A})$. Thus $I(f_C/\phi_C) \sim (I(f/\phi))_C$, and consequently

$$KF(I(f_C/\phi_C)) \sim KF((I(f/\phi))_C).$$

□

Theorem 24. *Let \mathcal{A} be an abelian category with enough injectives. Let \mathcal{Q} the category generated by a finite quiver Q , and assume that $RF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ is an equivalence of categories. Then $R(F_Q) : D^+(\mathcal{Q}(\mathcal{A})) \rightarrow D^+(\mathcal{Q}(\mathcal{B}))$ is also an equivalence of categories.*

Remark 25. *The finiteness of the quiver Q is a sufficient condition for $\mathcal{Q}(\mathcal{A})$ to have enough injectives. Alternatively, it may be replaced by the completeness of the category \mathcal{A} .*

Proof. Since, by the hypotheses, RF is an equivalence, then, as a consequence of the Proposition 11, $(RF)_{\mathcal{Q}}$ is also an equivalence, and hence is full and faithful. Furthermore, $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$ are also full and faithful. Then, by the commutativity of the diagram (22), we have that $R(F_{\mathcal{Q}})$ is full and faithful as well. Therefore, in order to complete the proof, it is sufficient to prove that $R(F_{\mathcal{Q}})$ is essentially surjective.

Indeed, set $B^{\bullet} \in \text{Ob}(D^+(\mathcal{Q}(\mathcal{B})))$; using the equivalence $K_{\mathcal{B}}$, we have the functor $B : \mathcal{Q} \longrightarrow \text{Kom}^+(\mathcal{B})$.

Then, for each $i \in \text{Ob}(\mathcal{Q})$, $B(i) \in \text{Ob}(D^+(\mathcal{B}))$. As RF is essentially surjective, there exist an object A_i^{\bullet} of $D^+(\mathcal{A})$ such that $RF(A_i^{\bullet}) \simeq B(i)$, that is, $KF(I(A_i^{\bullet})) \simeq B(i)$ in $D^+(\mathcal{B})$.

Taking a morphism $i \xrightarrow{p} j$ in \mathcal{Q} we have a morphism $B(i) \xrightarrow{B(p)} B(j)$ in $\text{Kom}^+(\mathcal{B})$, which yields a morphism

$$\begin{array}{ccc} & B(i) & \\ Id \swarrow & & \searrow [B(p)] \\ B(i) & & B(j) \end{array}$$

in $D(\mathcal{B})$. Since $\text{Hom}_{D(\mathcal{B})}(B(i), B(j)) \simeq \text{Hom}_{D(\mathcal{B})}(KF(I(A_i^{\bullet})), KF(I(A_j^{\bullet})))$, there exist a morphism $[g/\psi] \in \text{Hom}_{D(\mathcal{B})}(KF(I(A_i^{\bullet})), KF(I(A_j^{\bullet})))$ corresponding to $[B(p)/Id]$.

Since RF is full and faithful $RF : \text{Hom}_{D(\mathcal{A})}(A_i^{\bullet}, A_j^{\bullet}) \longrightarrow \text{Hom}_{D(\mathcal{B})}(KF(I(A_i^{\bullet})), KF(I(A_j^{\bullet})))$ is bijective, thus it follows that there is an unique $[f/\phi] \in \text{Hom}_{D(\mathcal{A})}(A_i^{\bullet}, A_j^{\bullet})$ such that $RF([f/\phi]) = [g/\psi]$ or $Q \circ KF(I([f/\phi])) = [g/\psi]$.

Our goal is to define a functor between \mathcal{Q} and $\text{Kom}^+(\mathcal{A})$. As $I([f/\phi])$ is a morphism in $K^+(\mathcal{A})$, we can choose a representative $A(p)$ in $\text{Kom}^+(\mathcal{A})$ for it, that is, $[A(p)] = I([f/\phi])$. Let us define the following functor

$$\begin{array}{ccc} A : \mathcal{Q} & \longrightarrow & \text{Kom}^+(\mathcal{A}) \\ i & \longmapsto & A(i) = A_i^{\bullet} \\ p \downarrow & & \downarrow A(p) \\ j & & A(j) = A_j^{\bullet}. \end{array}$$

Since $K_{\mathcal{A}}$ is an equivalence, there exists $A^{\bullet} \in \text{Ob}(D^+(\mathcal{Q}(\mathcal{A})))$ such that $K_{\mathcal{A}}(A^{\bullet}) = A$. We need to prove that $R(F_{\mathcal{Q}})(A^{\bullet}) \simeq B^{\bullet}$. Indeed, this means that we need to find quasi-isomorphisms $R(F_{\mathcal{Q}})(A^{\bullet}) \xleftarrow{\alpha} C^{\bullet} \xrightarrow{\beta} B^{\bullet}$.

By definition, $K(F_{\mathcal{Q}})(I(A^{\bullet})) = (F \circ I(A^{\bullet}))^{\bullet}$; then, for each $i \in \text{Ob}(\mathcal{Q})$, we have

$$K(F_{\mathcal{Q}})(I(A^{\bullet}))(i) = (F \circ I(A^{\bullet}))(i)^{\bullet} = KF(I(A^{\bullet}))(i).$$

As stated previously, $KF(I(A^{\bullet}))(i) \simeq KF(I(A(i)))$ in $K(\mathcal{B})$. On the other hand, $KF(I(A(i))) \sim B(i)$ in $D(\mathcal{B})$, so there are quasi-isomorphisms

$$(R(F_{\mathcal{Q}})(A^{\bullet}))(i) \xleftarrow{\alpha_i} C_i \xrightarrow{\beta_i} B(i).$$

Given $i \xrightarrow{p} j$ we have the diagram:

$$\begin{array}{ccc}
 & C_i & \overset{C_p}{\dashrightarrow} C_j \\
 \alpha_i \swarrow & & \searrow \alpha_j \\
 R(F_{\mathcal{Q}})(A^{\bullet})(i) & \xrightarrow{R(F_{\mathcal{Q}})(A^{\bullet})(p)} & R(F_{\mathcal{Q}})(A^{\bullet})(j) \\
 \beta_i \swarrow & & \searrow \beta_j \\
 B(i) & \xrightarrow{B(p)} & B(j),
 \end{array}$$

where the existence of C_p is ensured by the Lemma 18. Therefore, using the same arguments as in the proof of Theorem 21, we have the quasi-isomorphisms

$$\begin{array}{ccc}
 & C^{\bullet} & \\
 \alpha \swarrow & & \searrow \beta \\
 R(F_{\mathcal{Q}})(A^{\bullet}) & & B^{\bullet},
 \end{array}$$

as desired. \square

Finally, note that Lemma 23 and Theorem 24 also hold when we substitute D^+ for D^b . Namely, we can ensure the commutativity of the diagram

$$\begin{array}{ccc}
 D^b(\mathcal{C}(\mathcal{A})) & \xrightarrow{R(F_{\mathcal{C}})} & D^b(\mathcal{C}(\mathcal{B})) \\
 T_{\mathcal{A}} \downarrow & & \downarrow T_{\mathcal{B}} \\
 \mathcal{C}(D^b(\mathcal{A})) & \xrightarrow{(RF)_{\mathcal{C}}} & \mathcal{C}(D^b(\mathcal{B}))
 \end{array}$$

and, under the same hypothesis as Theorem 24, one can also show that if $RF : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{B})$ is an equivalence of categories, then $R(F_{\mathcal{Q}}) : D^b(\mathcal{Q}(\mathcal{A})) \rightarrow D^b(\mathcal{Q}(\mathcal{B}))$ is also an equivalence.

These results are due to the fact that, for every abelian category \mathcal{A} , $D^b(\mathcal{A})$ is equivalent to a full subcategory of $D^+(\mathcal{A})$.

5. FOURIER-MUKAI TRANSFORM OF \mathcal{Q} -SHEAVES

In this last Section of the paper, we will concentrate on categories of sheaves on algebraic varieties; more precisely, let X be a noetherian, separated scheme of finite type over an algebraically closed field \mathbb{K} , and let $\mathcal{Q}\mathcal{C}\mathcal{o}(X)$ and $\mathcal{C}\mathcal{o}\mathcal{h}(X)$ denote the categories of quasi-coherent and coherent sheaves on X , respectively.

We denote by $D(X)$ the derived category of complexes of quasi-coherent sheaves with coherent cohomology, i.e. $D(X) := D_{\mathcal{C}\mathcal{o}\mathcal{h}(X)}(\mathcal{Q}\mathcal{C}\mathcal{o}(X))$. The derived categories $D^*(X)$ with $* = b, -, +$ are defined analogously. Recall that $D^b(X)$ is equivalent to $D^b(\mathcal{C}\mathcal{o}\mathcal{h}(X))$.

5.1. Derived categories of \mathcal{Q} -sheaves. Given a quiver \mathcal{Q} , we define a *quasi-coherent \mathcal{Q} -sheaf* on X as a functor from $\mathcal{Q} \rightarrow \mathcal{Q}\mathcal{C}\mathcal{o}(X)$, and we denote by $\mathcal{Q}\mathcal{Q}(X)$ the category $\mathcal{Q}(\mathcal{Q}\mathcal{C}\mathcal{o}(X))$ of the quasi-coherent \mathcal{Q} -sheaves on X . Similarly, we define a *coherent \mathcal{Q} -sheaf* on X as a functor from $\mathcal{Q} \rightarrow \mathcal{C}\mathcal{o}\mathcal{h}(X)$, and we denote by $\mathcal{Q}\mathcal{C}(X)$ the category $\mathcal{Q}(\mathcal{C}\mathcal{o}\mathcal{h}(X))$ of the coherent \mathcal{Q} -sheaves on X ; cf. [1, 4]. Several types of

sheaves with additional structure (or *decorated sheaves*) considered in the literature, like Higgs bundles, parabolic bundles, coherent systems and holomorphic triples, can be regarded as Q -sheaves.

Lemma 26. *The categories $\mathcal{Q}Q(X)$ and $\mathcal{Q}C(X)$ are abelian. Moreover, if Q is a finite quiver then $\mathcal{Q}Q(X)$ has enough injectives.*

Proof. The first claim stems of the fact that the categories $\mathcal{Q}\mathbf{co}(X)$ and $\mathcal{C}\mathbf{oh}(X)$ are abelian so, using the Proposition 2, we have that $\mathcal{Q}Q(X)$ and $\mathcal{Q}C(X)$ are also abelian.

Since X is a noetherian scheme, $\mathcal{Q}\mathbf{co}(X)$ has enough injectives. Therefore, if Q is a finite quiver, by Corollary 5, we have that $\mathcal{Q}Q(X)$ also has enough injectives. \square

Now in order to properly define the derived categories of coherent Q -sheaves we will need the following lemma.

Lemma 27. *Let \mathcal{A}' be a thick abelian subcategory of \mathcal{A} . Then $\mathcal{C}(\mathcal{A}')$ is a thick abelian subcategory of $\mathcal{C}(\mathcal{A})$.*

Proof. Let F and G be objects of $\mathcal{C}(\mathcal{A}')$, thus in particular also objects of $\mathcal{C}(\mathcal{A})$. Let H be another object in $\mathcal{C}(\mathcal{A})$ such that the following sequence is exact:

$$(23) \quad 0 \longrightarrow F \longrightarrow H \longrightarrow G \longrightarrow 0.$$

For each morphism $C \xrightarrow{t} D$ in \mathcal{C} we have the commutative diagram

$$(24) \quad \begin{array}{ccccccc} 0 & \longrightarrow & F(C) & \longrightarrow & H(C) & \longrightarrow & G(C) \longrightarrow 0 \\ & & \downarrow F(t) & & \downarrow H(t) & & \downarrow G(t) \\ 0 & \longrightarrow & F(D) & \longrightarrow & H(D) & \longrightarrow & G(D) \longrightarrow 0 \end{array}$$

in \mathcal{A} , where $F(C)$, $G(C)$, $F(D)$ and $G(D)$ are objects of \mathcal{A}' . As \mathcal{A}' is thick, $H(C)$ and $H(D)$ are objects of \mathcal{A}' . But \mathcal{A}' is a full subcategory and then $H(t)$ is a morphism in \mathcal{A}' . So, the diagram (24) is in \mathcal{A}' , then the sequence (23) is in $\mathcal{C}(\mathcal{A}')$.

Moreover, note that $\mathcal{C}(\mathcal{A}')$ is a full subcategory of $\mathcal{C}(\mathcal{A})$. Indeed, set F and G in $Ob(\mathcal{C}(\mathcal{A}'))$ and $\eta \in Hom_{\mathcal{C}(\mathcal{A})}(F, G)$. For each $C \in Ob(\mathcal{C})$, note that η_C belongs to $Hom_{\mathcal{A}}(F(C), G(C)) = Hom_{\mathcal{A}'}(F(C), G(C))$. It follows that $\eta \in Hom_{\mathcal{C}(\mathcal{A}')} (F, G)$. \square

As a consequence of this Lemma, $\mathcal{Q}C(X)$ is a thick abelian subcategory of $\mathcal{Q}Q(X)$, and we then define:

$$D(\mathcal{Q}(X)) := D_{\mathcal{Q}C(X)}(\mathcal{Q}Q(X)).$$

Similarly, we define $D^+(\mathcal{Q}(X))$, $D^-(\mathcal{Q}(X))$ and $D^b(\mathcal{Q}(X))$. Our next result generalizes the well-known equivalence for the bounded derived category of sheaves on an algebraic variety.

Proposition 28. *If X is an algebraic variety, one has an equivalence of categories*

$$D^b(\mathcal{Q}(X)) \simeq D^b(\mathcal{Q}C(X)).$$

Proof. We argue that the inclusion functor $i : DD^b(\mathcal{Q}C(X)) \longrightarrow DD^b_{\mathcal{Q}C(X)}(\mathcal{Q}Q(X))$ is an equivalence.

Let us first prove that i is essentially surjective. Indeed, let $R^\bullet \in D_{\mathcal{QC}(X)}^b(\mathcal{QQ}(X))$, we want to find $G^\bullet \in D^b(\mathcal{QC}(X))$ such that $i(G^\bullet) \simeq R^\bullet$, that is, $G^\bullet \simeq R^\bullet$ in $D_{\mathcal{QC}(X)}^b(\mathcal{QQ}(X))$.

Using the restriction of the functor K of the Lemma 13 we can consider for each $i \in \text{Ob}(\mathcal{Q})$, $R(i)^\bullet \in \text{Ob}(D_{\mathfrak{Coh}(X)}(\mathfrak{QCoh}(X)))$. By [11, Lemma II.1] exist $G_C \in \text{Ob}(D(\mathfrak{Coh}(X)))$ and a quasi-isomorphism $\alpha_C : G_C \rightarrow R(i)^\bullet$.

Therefore, given a morphism $i \xrightarrow{p} j$ in \mathcal{Q} , we have:

$$(25) \quad \begin{array}{ccc} & G_i & \xrightarrow{G_p} G_j \\ \alpha_i \swarrow & & \searrow \alpha_j \\ R(i)^\bullet & \xrightarrow{R(p)} & R(j)^\bullet \\ \alpha_i \swarrow & & \searrow \alpha_j \\ R(i)^\bullet & \xrightarrow{R(p)} & R(j)^\bullet \end{array}$$

where G_p is given by Lemma 18 and using the same arguments of Theorem 21 we define:

$$\begin{array}{ccc} G : \mathcal{Q} & \longrightarrow & \text{Kom}^+(\mathfrak{Coh}(X)) \\ i & \longmapsto & G(i) = G_i \\ p \downarrow & & \downarrow G(p)=G_p \\ j & & G(j) = G_j. \end{array}$$

Again, by equivalence K we have $G^\bullet \in \text{Ob}(D(\mathcal{QC}(X)))$ and α is a quasi-isomorphism between R^\bullet and G^\bullet , which completes the proof that i is essentially surjective.

Finally, we prove that i is fully faithful.

Let R^\bullet and H^\bullet objects in $D(\mathcal{QC}(X))$. A morphism $R^\bullet \xrightarrow{p} H^\bullet$ in $D_{\mathcal{QC}(X)}(\mathcal{QQ}(X))$ can be represented by

$$\begin{array}{ccc} & S^\bullet & \\ \phi \swarrow & & \searrow f \\ R^\bullet & & H^\bullet, \end{array}$$

Then, as i is essentially surjective, there exist G^\bullet in $D(\mathcal{QC}(X))$ and a quasi-isomorphism $G^\bullet \xrightarrow{\alpha} S^\bullet$ then

$$\begin{array}{ccc} & G^\bullet & \\ \phi \circ \alpha \swarrow & & \searrow f \circ \alpha \\ R^\bullet & & H^\bullet, \end{array}$$

define a morphism between R^\bullet and H^\bullet in $D(\mathcal{QC}(X))$. Therefore i is fully faithful. \square

5.2. Integral functors and Fourier-Mukai transforms for quiver sheaves.

Recall that given projective varieties X and Y over \mathbb{K} and an object $\mathcal{K}^\bullet \in D^b(X \times$

Y), an *integral functor* with kernel \mathcal{K}^\bullet is the functor $\Phi^{\mathcal{K}^\bullet} : D^b(X) \rightarrow D^b(Y)$ given by

$$\Phi^{\mathcal{K}^\bullet}(\mathcal{E}^\bullet) := R\pi_{Y*}(\pi_X^* \mathcal{E}^\bullet \otimes^L \mathcal{K}^\bullet) .$$

If $\Phi^{\mathcal{K}^\bullet}$ is an exact equivalence of derived categories, then it is called a *Fourier-Mukai functor*. Additionally, if the kernel $\mathcal{K}^\bullet = \mathcal{K}$ is a concentrated complex, i.e. a sheaf, then $\Phi^{\mathcal{K}}$ is called a *Fourier-Mukai transform*.

Furthermore, recall also that if the kernel $\mathcal{K}^\bullet = \mathcal{K}$ is a locally free sheaf, then $\Phi^{\mathcal{K}}$ is the right derived functor of the functor $\phi^{\mathcal{K}} : \mathfrak{Coh}(X) \rightarrow \mathfrak{Coh}(Y)$ given by

$$\phi^{\mathcal{K}}(\mathcal{E}) := \pi_{Y*}(\pi_X^* \mathcal{E} \otimes \mathcal{K}) .$$

Similarly, one may consider integral functors for Q -sheaves. More precisely, let \mathcal{K}^\bullet be an object of $D^b(\mathcal{QC}(X \times Y))$. Consider the integral functor $\Psi^{\mathcal{K}^\bullet} : D^b(\mathcal{QC}(X)) \rightarrow D^b(\mathcal{QC}(Y))$ given by

$$\Psi^{\mathcal{K}^\bullet}(\mathcal{E}^\bullet) := R\pi_{Y*}(\pi_X^* \mathcal{E}^\bullet \otimes^L \mathcal{K}^\bullet) ,$$

with the tensor product between Q -sheaves being taken vertex-by-vertex and arrow-by-arrow; more precisely:

$$(\{\mathcal{E}_v\}, \{\phi_a\}) \otimes (\{\mathcal{E}'_v\}, \{\phi'_a\}) := (\{\mathcal{E}_v \otimes \mathcal{E}'_v\}, \{\phi_a \otimes \phi'_a\}) .$$

One simple way to construct integral functors for Q -sheaves is taking the right derived of the induced functor $\phi_Q^{\mathcal{K}} : \mathcal{QC}(X) \rightarrow \mathcal{QC}(Y)$, which yields an integral functor $R(\phi_Q^{\mathcal{K}}) : D^b(\mathcal{QC}(X)) \rightarrow D^b(\mathcal{QC}(Y))$. In comparison with the general integral functors considered in the previous paragraph, the functor $R(\phi_Q^{\mathcal{K}})$ is indeed the particular case where \mathcal{K}^\bullet is the Q -sheaf (i.e. concentrated Q -complex) in which all vertices are decorated with the same coherent sheaf \mathcal{K} on $X \times Y$, and all arrows are decorated with the identity map.

The natural problem that arises is to characterize when an integral functor $\Psi^{\mathcal{K}^\bullet} : D^b(\mathcal{QC}(X)) \rightarrow D^b(\mathcal{QC}(Y))$ yields an equivalence of categories, i.e. are Fourier-Mukai functors.

In the following section, we will show that, under certain hypothesis, functors of the form $R(\phi_Q^{\mathcal{K}})$ are actually Fourier-Mukai transforms for Q -sheaves on abelian varieties.

5.3. A Mukai Theorem for Q -sheaves on abelian varieties. Now let X denote an abelian variety and Y its dual abelian variety, and consider the integral functor $\mathcal{S} : \mathfrak{Qco}(X) \rightarrow \mathfrak{Qco}(Y)$ defined by $\mathcal{S}(\mathcal{E}) := \pi_{Y*}(\pi_X^* \mathcal{E} \otimes \mathcal{P})$, where \mathcal{P} is the Poincaré line bundle over $X \times Y$. Mukai has proved [7] that its derived functor $R\mathcal{S} = \Phi_{X \rightarrow Y}^{\mathcal{P}} : D^b(\mathfrak{Qco}(X)) \rightarrow D^b(\mathfrak{Qco}(Y))$, is an equivalence of categories. The same is true for the functors acting on coherent sheaves.

In this Section, we show that the induced functor \mathcal{S}_Q is an also a derived equivalence of categories. Indeed, the following result for quasi-coherent Q -sheaves follows immediately from Theorem 24.

Corollary 29. *Let \mathcal{Q} be the category generated by a finite quiver Q . The integral functor*

$$R(\mathcal{S}_Q) : D^b(\mathcal{QQ}(X)) \rightarrow D^b(\mathcal{QQ}(Y))$$

is an equivalence of categories.

Our goal now is to prove that the same functor also provides an equivalence between $D^b(\mathcal{QC}(X))$ and $D^b(\mathcal{QC}(Y))$; note that this does not follow from Theorem 24 because the category $\mathfrak{Ch}(X)$ does not have enough injectives.

To go around this difficulty, let \mathcal{A}' be a thick abelian subcategory of \mathcal{A} , by Lemma 27, $\mathcal{C}(\mathcal{A}')$ is a thick abelian subcategory of $\mathcal{C}(\mathcal{A})$ then we can define the subcategory $Kom_{\mathcal{C}(\mathcal{A}')}(\mathcal{C}(\mathcal{A}))$ of $Kom(\mathcal{C}(\mathcal{A}))$ of the complexes in $\mathcal{C}(\mathcal{A})$ whose cohomology objects are $\mathcal{C}(\mathcal{A}')$. Note that the functor K , defined in the Lemma 13, restricted to $Kom_{\mathcal{C}(\mathcal{A}')}(\mathcal{C}(\mathcal{A}))$ has its image in $\mathcal{C}(Kom_{\mathcal{A}'}(\mathcal{A}))$ and is an isomorphism between $Kom_{\mathcal{C}(\mathcal{A}')}(\mathcal{C}(\mathcal{A}))$ and $\mathcal{C}(Kom_{\mathcal{A}'}(\mathcal{A}))$. This fact is due to $H^i(F^\bullet)(C) = H^i(F(C)^\bullet)$ for all $F^\bullet \in Ob(Kom(\mathcal{C}(\mathcal{A})))$ and for all $C \in Ob(\mathcal{C})$.

Therefore we can define

$$T_{\mathcal{AA}'} = T_{\mathcal{A}}|_{D_{\mathcal{C}(\mathcal{A}')}(\mathcal{C}(\mathcal{A}))} : D_{\mathcal{C}(\mathcal{A}')}(\mathcal{C}(\mathcal{A})) \longrightarrow \mathcal{C}(D_{\mathcal{A}'}(\mathcal{A}))$$

Note that $T_{\mathcal{AA}'}(F^\bullet)$ is actually an object of $\mathcal{C}(D_{\mathcal{A}'}(\mathcal{A}))$ by the new approach of functor K . Moreover, if $\mathcal{C} = \mathcal{Q}$, where \mathcal{Q} is the category generated by a finite quiver Q , then $T_{\mathcal{AA}'}$ is also fully faithful.

Let \mathcal{B}' be a thick abelian subcategory of \mathcal{B} and let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor, such that its extension $RF : D_{\mathcal{A}'}(\mathcal{A}) \longrightarrow D_{\mathcal{B}'}(\mathcal{B})$ is well defined whenever \mathcal{A} has enough injectives. We would like to define $R(F_{\mathcal{C}}) : D_{\mathcal{C}(\mathcal{A}')}(\mathcal{C}(\mathcal{A})) \longrightarrow D_{\mathcal{C}(\mathcal{B}')}(\mathcal{C}(\mathcal{B}))$, that is, given A^\bullet an object in $D_{\mathcal{C}(\mathcal{A}')}(\mathcal{C}(\mathcal{A}))$, we would like that $R(F_{\mathcal{C}})(A^\bullet) \in Ob(D_{\mathcal{C}(\mathcal{B}')}(\mathcal{C}(\mathcal{B})))$. For this we need to prove that $H^i(R(F_{\mathcal{C}})(A^\bullet)) \in Ob(\mathcal{C}(\mathcal{B}'))$. Indeed, by Proposition 16, $H^i(R(F_{\mathcal{C}})(A^\bullet))(C) = H^i(R(F_{\mathcal{C}})(A^\bullet)(C))$ for all $C \in Ob(\mathcal{C})$, and

$$R(F_{\mathcal{C}})(A^\bullet)(C) = K(F_{\mathcal{C}})(I(A^\bullet))(C) = KF(I(A^\bullet)(C)), \forall C \in Ob(\mathcal{C}).$$

Moreover, $I(A^\bullet)$ is quasi-isomorphic to A^\bullet then $I(A^\bullet)(C)$ is an object in $D_{\mathcal{A}'}(\mathcal{A})$ therefore, as $RF : D_{\mathcal{A}'}(\mathcal{A}) \longrightarrow D_{\mathcal{B}'}(\mathcal{B})$ is well defined, $KF(I(A^\bullet)(C)) \in Ob(D_{\mathcal{B}'}(\mathcal{B}))$ and consequently $H^i(R(F_{\mathcal{C}})(A^\bullet))(C) = H^i(KF(I(A^\bullet)(C)))$ is an object in \mathcal{B}' , for all $C \in Ob(\mathcal{C})$. Finally $H^i(R(F_{\mathcal{C}})(A^\bullet)) \in Ob(\mathcal{C}(\mathcal{B}'))$.

Accordingly, and follow the proof of the Lemma 23, we have the following commutative diagram:

$$\begin{array}{ccc} D_{\mathcal{C}(\mathcal{A}')}^+(\mathcal{C}(\mathcal{A})) & \xrightarrow{R(F_{\mathcal{C}})} & D_{\mathcal{C}(\mathcal{B}')}^+(\mathcal{C}(\mathcal{B})) \\ T_{\mathcal{AA}'} \downarrow & & \downarrow T_{\mathcal{BB}'} \\ \mathcal{C}(D_{\mathcal{A}'}^+(\mathcal{A})) & \xrightarrow{(RF)_c} & \mathcal{C}(D_{\mathcal{B}'}^+(\mathcal{B})), \end{array}$$

Furthermore, we have the following result; its proof is analogous to the proof of Theorem 24:

Theorem 30. *Let \mathcal{A} be an abelian category with enough injectives, \mathcal{A}' be a thick abelian subcategory of \mathcal{A} , \mathcal{B}' be a thick abelian subcategory of \mathcal{B} and let \mathcal{Q} be a category generated by a finite quiver Q . If $RF : D_{\mathcal{A}'}^+(\mathcal{A}) \longrightarrow D_{\mathcal{B}'}^+(\mathcal{B})$ is an equivalence of categories then $R(F_{\mathcal{Q}}) : D_{\mathcal{Q}(\mathcal{A}')}^+(\mathcal{Q}(\mathcal{A})) \longrightarrow D_{\mathcal{Q}(\mathcal{B}')}^+(\mathcal{Q}(\mathcal{B}))$ is also an equivalence of categories.*

As also noted at the end of the previous Section, the same result is valid substituting D^+ for D^b : if $RF : D_{\mathcal{A}'}^b(\mathcal{A}) \longrightarrow D_{\mathcal{B}'}^b(\mathcal{B})$ is an equivalence of categories then $R(F_{\mathcal{Q}}) : D_{\mathcal{Q}(\mathcal{A}')}^b(\mathcal{Q}(\mathcal{A})) \longrightarrow D_{\mathcal{Q}(\mathcal{B}')}^b(\mathcal{Q}(\mathcal{B}))$ is also an equivalence of categories.

As an application of our last Theorem, we have:

Corollary 31. *Let \mathcal{Q} be the category generated by a finite quiver Q . The integral functors*

$$R(S_Q) : D^b(\mathcal{Q}(X)) \rightarrow D^b(\mathcal{Q}(Y)) \quad \text{and} \quad R(S_Q) : D^b(\mathcal{Q}C(X)) \rightarrow D^b(\mathcal{Q}C(Y))$$

are equivalences of categories.

Proof. As previously stated, we have that $\Phi^{\mathcal{K}} : D^b(X) \rightarrow D^b(Y)$ is a Fourier-Mukai transform. Then, by the Theorem 30

$$R(S_Q) : D^b(\mathcal{Q}(X)) \rightarrow D^b(\mathcal{Q}(Y))$$

is an equivalence. The second equivalence follows immediately from Proposition 28. \square

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